Gribov copies: thermodynamical and continuum limits

V. G. Bornyakov[1], V. K. Mitrjushkin[2], and R. N. Rogalyov[1]

1. Institute for High-Energy Physics
2. Joint Insitute of Nuclear Research

02.04.2013

- Problem of Gribov copies: introduction
- First Gribov horizon
- Fundamental modular region
- Gauge fixing algorithm on a lattice
- The center symmetry
- Momentum dependence of the propagator
- The effect of Gribov copies
- The effects of finite volume

The gauge-field action

$$S = -\frac{1}{4}\int d^D x \; F^a_{\mu\nu} F^a_{\mu\nu},$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

is invariant under $SU(N)$ transformations $\Lambda = \exp\left(-i\omega^a \Gamma^a\right)$:

$$A_\mu \to A^\Lambda_\mu = \Lambda^\dagger A_\mu \Lambda + \frac{i}{g}\,\Lambda^\dagger \partial_\mu \Lambda. \tag{1}$$

To exclude nonphysical degrees of freedom, we impose the gauge conditions:

$$\Phi(A) = 0, \qquad \Phi(A^\Lambda) \neq 0 \quad \forall \Lambda : \Lambda \neq 1. \tag{2}$$

$$A_\mu \to A_\mu^\Lambda = (\Lambda Z)^\dagger A_\mu (\Lambda Z) + \frac{i}{g} (\Lambda Z)^\dagger \partial_\mu (\Lambda Z).$$

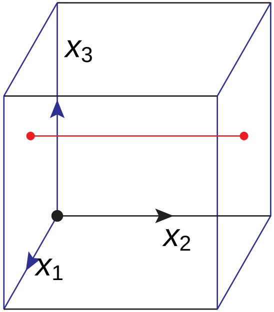
$$A_\mu \to A_\mu^\Lambda = \Lambda^\dagger A_\mu \Lambda + \frac{i}{g} \Lambda^\dagger \partial_\mu \Lambda.$$

For $SU(3)$, as an example:

$$Z \in \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{\frac{2i\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{2i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{2i\pi}{3}} \end{pmatrix}, \begin{pmatrix} e^{\frac{4i\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{4i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{4i\pi}{3}} \end{pmatrix} \right\}$$

Gauge transformation is the same on both sides!

1. We consider the $SU(2)$ theory in 3 dimensions, so we need IR and UV regularizations.
2. We put the system in a box of size b.
3. Gauge-invariant UV regularization is provided by a lattice.



We extend the gauge group by nonperiodic gauge transformations:

$$\Lambda(x_1, b, x_3) = Z\Lambda(x_1, 0, x_3) \text{ etc.}$$

$$P\exp\left(ig\int_0^b A_2(x_1, z, x_3)dz\right) =$$

$$= L(x_1, x_3) \longrightarrow L(x_1, x_3)Z$$

Thus the Hilbert space is broken into 8 superselection sectors

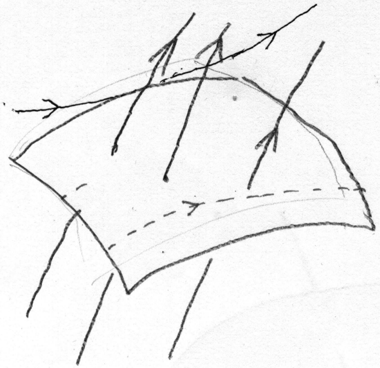
We consider the Landau gauge $\partial_\mu A_\mu = 0$.
Note that

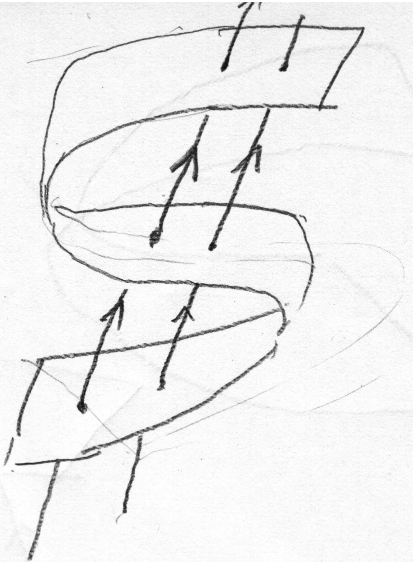
$$\partial_\mu A_\mu^\Lambda = \frac{i}{g} D_\mu \xi_\mu + \Lambda^\dagger (\partial_\mu A_\mu) \Lambda, \tag{3}$$

where

$$D_\mu \phi = \partial_\mu \phi - ig \left[A_\mu, \phi \right], \qquad \xi_\mu = \Lambda^\dagger \partial_\mu \Lambda. \tag{4}$$

- From (3) it follows that the condition $\partial_\mu A_\mu = 0$ is invariant under constant gauge transformation. Thus we consider the gauge transformations $\Lambda(x) \in \bar{\mathcal{G}} = \mathcal{G}/G$.
- In the class of “large” gauge transformations, (3) has nontrivial solutions (Gribov, 1978)





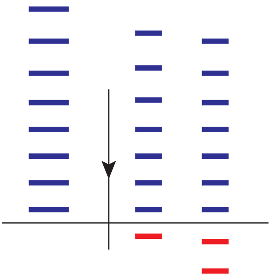
If $\Lambda \simeq 1 + \omega$, $\omega \to 0$ then

$$\partial_\mu A_\mu^\Lambda - \partial_\mu A_\mu = \frac{i}{g}\,\partial_\mu D_\mu \omega$$

$$\equiv -\frac{i}{g}\,M_{FP}(A)\,\omega. \qquad (5)$$

For sufficiently small A_μ, gauge variation $\delta_\omega(\partial_\mu A_\mu)$ does not vanish because the minimal eigenvalue of $M_{FP}(A = 0)$ is $\lambda_{min}^{FP} = \frac{4\pi^2}{b^2}$.

Eigenvalues of $M_{FP}(\alpha_n B)$:



$\alpha_1 \;<\; \alpha_2 \;<\; \alpha_3$

Let us consider the field $B_\mu \neq 0$ and the line $\{\alpha B\}$, $\alpha > 0$

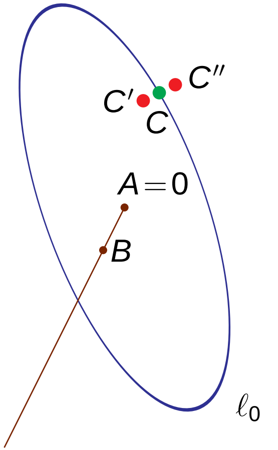
First Gribov horizon

$$\ell_0 = \{A : \partial_\mu A_\mu = 0,\ \lambda^{FP}_{min} = 0\}$$

$\ell_0 = \partial\Omega_0$, where Ω_0
is the first Gribov region.

$$\forall C \in \ell_0\ \exists \phi_0 :\ M_{FP}(C)\phi_0 = 0.$$
$$\Rightarrow\ \delta_\omega(\partial_\mu C_\mu) \simeq O(\omega^2)$$

$$\forall C' \in \Omega_0 : \|C' - C\| \simeq \epsilon$$
$$\exists C'' = C' + \delta_{\epsilon\phi_0} C' \in \bar{\Omega}_0$$



Semenov-TyanShanskii–Franke functional

$$\mathcal{F}(A) = \int dx\, A_\mu^a(x) A_\mu^a(x)$$

Fundamental modular region:

$$\Gamma = \{A : \mathcal{F}(A) \leq \mathcal{F}(A^\Lambda) \quad \forall \Lambda \in \mathcal{G}\}$$

$$\mathcal{F}(A^\Lambda) - \mathcal{F}(A) = -\frac{1}{g^2} \int dx \left\langle \Lambda^\dagger \Box \Lambda - 2igA_\mu(\partial_\mu \Lambda)\Lambda^\dagger \right\rangle$$

Γ is convex!

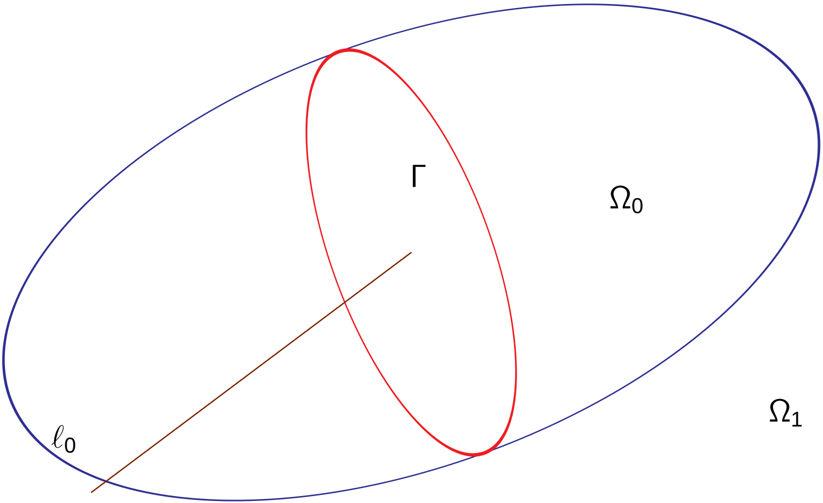
$$\Lambda \simeq 1 + \omega + \omega^2/2 + ...$$

$$\begin{aligned} \delta_\omega \mathcal{F}(A) &= 2\int_M dx \omega^a(x) \partial_\mu A^a_\mu(x) + \\ &+ \int_M dx\, dy\, \omega^a(x) M^{ab}_{FP}(A; x, y) \omega^b(y) + ... \end{aligned} \tag{6}$$

Extremum of $\mathcal{F}$ gives a configuration A^{extr} satisfying

$$\partial_\mu A^{extr}_\mu = 0,$$

if $A^{extr} \in \Omega_0$, the extremum is minimum.


Γ
Ω0
Ω1
ℓ0

The Gribov-Zwanziger effective action

- Integration over the first Gribov region

$$Z = \int DA\ \delta(\partial A)\ \theta(A \in \Omega_0)\,\mathrm{det}[M_{FP}(A)]\,\exp[-S_{YM}(A)]$$

- Finite volume; UV cutoff $\simeq \kappa$

$$S_{eff} = \int_M dx\ A^a_\mu(x)\ \left(\Box + \frac{\kappa^4}{\Box}\right)\left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\right) A^a_\nu(x)$$

Osterwalder-Schrader reflection positivity

Condition for the propagator in a scalar field theory:

$$\int dx\, dy\, f^{\dagger}(-x_4, \vec{x}) D(x-y) f(y_4, \vec{y}) \geq 0 \quad \forall f \tag{7}$$

If the Källen–Lehmann representation

$$D(p) = \int_0^{\infty} dm^2\, \frac{\rho(m^2)}{p^2 + m^2} \tag{8}$$

is valid, the OS positivity corresponds to positivity of $\rho(m^2)$

Infrared behavior of the gluon propagator in the Landau gauge
is of interest because

- Propagator is needed for calculation of physical quantities;
- The Kugo-Ojima and Gribov-Zwanziger
  confinement criteria
  are formulated in terms of propagator behavior
  in the Euclidean domain.

If the Osterwalder-Schrader reflection positivity
is violated for the gluon fields,
one cannot construct
the respective Hilbert space with positive metric.
The gluon fields are not associated with asymptotic states.

$\Longrightarrow$ gluons are confined

- It is of interest to compare lattice and continuum results (say, from the Schwinger–Dyson Equations) for the propagator
- Gauge fixing on a lattice is also of intererst because the respective continuum gauge theory is defined only in a particular gauge.

The gluon propagator in the Landau gauge:

$$D_{\mu\nu}^{ab}(p) = \delta^{ab} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) D(p)$$

The Functional Renormalization Group (FRG)
and the Schwinger–Dyson Equations (SDE)
imply at $p \to 0$ [Fischer, Pawlowsky, 2006; Alkofer etc]:

- scaling solution:

$$D(p) \simeq (p^2)^{2\kappa + (2-D)/2} \qquad G(p) \simeq (p^2)^{-1-\kappa}, \tag{9}$$

- massive solution

$$D(p) \simeq const \qquad G(p) \simeq \frac{Z}{(p^2)}, \tag{10}$$

- scaling solution:

$$D(p) \simeq p^{4\kappa + 2 - D} \qquad G(p) \simeq p^{-2-2\kappa}, \tag{11}$$

DSE & FRG imply

$$\begin{aligned} D = 2 : \kappa = 1/5 \qquad & D(p) \simeq p^{2/5}; \qquad & G(p) \simeq p^{-12/5}; \\ D = 3 : \kappa = 0.3975 \qquad & D(p) \simeq p^{0.59}; \qquad & G(p) \simeq p^{-2.8}; \\ D = 4 : \kappa = 0.595 \qquad & D(p) \simeq p^{0.4}; \qquad & G(p) \simeq p^{-3.2}; \end{aligned}$$

$\alpha(p) = D(p)G^2(p)p^{2+D} \simeq const$ as $p \to 0$

$$S = \frac{4}{g^2 a} \sum_{P=x,\mu,\nu} \left(1 - \frac{1}{2}\mathrm{Tr}\;U_P\right)$$

where

$$U_P = U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu}$$

$U_{x,\mu} \in SU(2)$, $D = 3$

$$U_{x,\mu} \;=\; u_0 \;+\; i \sum_{a=1}^{3} u_a \sigma_a, \qquad (12)$$

$$A^a_\mu = -\frac{2u^a_\mu}{ga}, \qquad (13)$$

$$\Lambda: \quad U_{x,\mu} \to \Lambda^\dagger_x U_{x,\mu} \Lambda_{x+\hat{\mu}},$$

We fix the absolute Landau gauge by finding the global maximum of the functional

$$\mathcal{F}[\,\mathcal{U}\,] = \frac{1}{2} \sum_{x,\mu} \mathrm{Tr}\, U_{x,\mu}, \qquad (14)$$

Stationarity condition:

$$\partial_\nu A^a_\nu = 0.$$

We study the gluon propagator

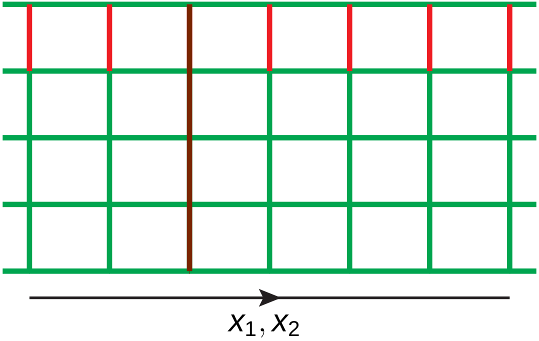
$$D^{bc}_{\mu\nu}(q) = \frac{a^3}{L^3} \sum_{x,y\in\Lambda} \exp(iqx)\langle A^b_\mu(x+y)A^c_\nu(y)\rangle, \tag{15}$$

where

$$\langle A^b_\mu(x+y)A^c_\nu(y)\rangle = \frac{1}{\mathcal{Z}} \int DUe^{-S[U]} A^b_\mu(x+y)A^c_\nu(y) \tag{16}$$

$$D^{bc}_{\mu\nu}(q) = \begin{cases} \delta^{bc}\delta_{\mu\nu}\bar{D}(0), & p = 0; \\ \delta^{bc}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\bar{D}(p), & p \neq 0, \end{cases}$$

where $p_\mu = \frac{2}{a}\sin\frac{q_\mu a}{2}$ and $p^2 = \sum_{\mu=1}^{3} p_\mu^2$.



Center symmetry:

$\mathbb{Z}_2 : U_{x,\mu} \to -U_{x,\mu}$

$L(x_1, x_2) \to -L(x_1, x_2)$

$$L(x_1, x_2) = Tr \prod_{j=1}^{N_\tau} U(x + j\hat{3}, 3)$$

We extend the gauge group

$$\mathcal{G} \longrightarrow \mathcal{G}_E = \mathcal{G} \times \mathbb{Z}_2^3, \tag{17}$$

where $\mathcal{G} = \{\Lambda(x)\}, \quad \Lambda(x) \in SU(2)$:

$$U_{x,\mu} \to \Lambda_x^\dagger U_{x,\mu} \Lambda_{x+\hat{\mu}}, . \tag{18}$$

The configuration space $\{\mathcal{U}\}$ is divided into 8 $\mathbb{Z}_2^3$ sectors, according to the signs of

$$\sum_{x_\mu=a}^{La} \sum_{x_\nu=a}^{La} L(x_\mu, x_\nu)$$

We consider the following versions of the Landau gauge:

- 'First-copy Landau gauge' (*fc*):
  choose arbitrary Gribov copy within Ω_0.
- Simulated Annealing Landau gauge (*SA*):
  choose the Gribov copy with the maximum value of F (maximum with respect to **periodic** gauge transformations).
- Flipped Simulated Annealing Landau gauge (*FSA*):
  choose the Gribov copy with the maximum value of F (maximum with respect to **both periodic and nonperiodic** gauge transformations).

$$D(p) \neq D(p) \neq D(p)!!!$$

Problem of degenerate maxima.

The simulated annealing (SA) algorithm
generates gauge transformations $\Lambda(x)$
by MC iterations with a statistical weight $\sim \exp\left(4V\, F[\Lambda]/T\right)$.
T is an auxiliary parameter which is gradually decreased
from $T_{\rm init} = 1.3$, to $T_{\rm final} = 0.01$ in order to maximize $F[\Lambda]$.
The final SA temperature is fixed so that the quantity

$$\max_{x,a} \left| \sum_{\mu=1}^{3} \left(A^a_{x+\hat{\mu}/2;\mu} - A^a_{x-\hat{\mu}/2;\mu} \right) \right| \qquad (19)$$

decreases monotonously during subsequent overrelaxation
(OR) for the majority of gauge-fixing trials.
The number of the SA steps is set equal to 3000.

We use the standard Los-Alamos type overrelaxation

The number of iterations:

$500 \div 700$ at $L = 32$ $1500 \div 3000$ at $L = 80$;

in few cases, several times greater.

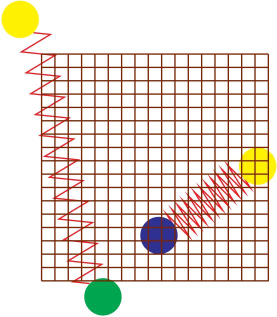
The precision of gauge fixing:

$$\max_{x,a} \left| \sum_{\mu=1}^{3} \left(A^a_{x+\hat{\mu}/2;\mu} - A^a_{x-\hat{\mu}/2;\mu} \right) \right| < 10^{-7} \qquad (20)$$

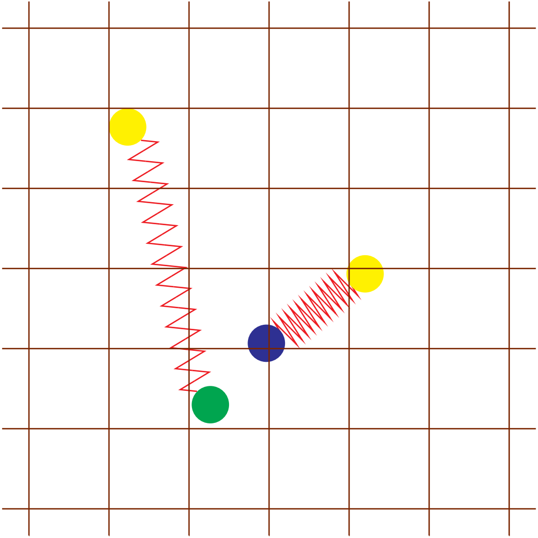
The configuration with the greatest value of $F[\Lambda]$ is referred to as “the best copy”.

- We repeat this procedure (SA and OR) $N_{meas} \simeq 1000$ times;
- Then we take an average over the measurements.

continuum limit $a \to 0$



Weak coupling



Strong coupling (TD limit, $b \to \infty$)

For $SU(2)$ theory in 3 dimensions, for $\beta = \frac{4}{ag^2} > 3$

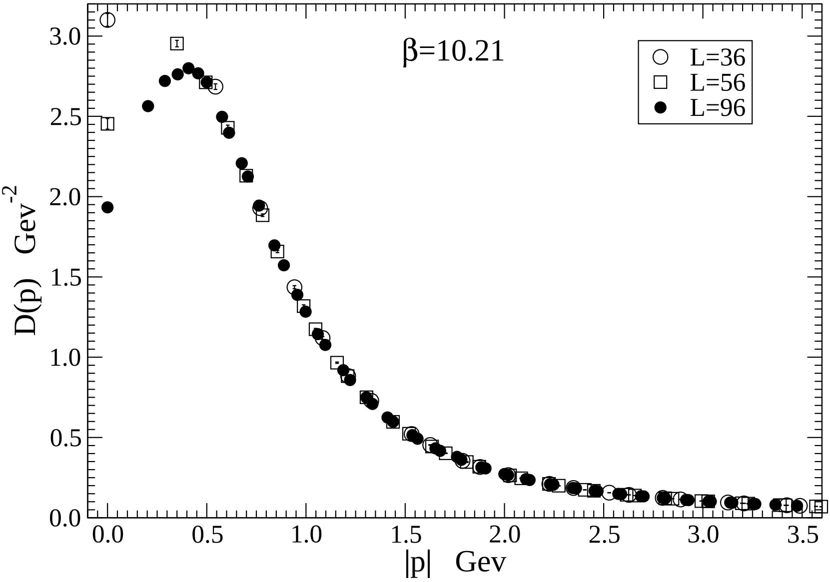
M.Teper [1998] obtained:

$$a\sqrt{\sigma} = \frac{1.337(23)}{\beta} + \frac{0.95(38)}{\beta^2} + \frac{1.1(1.3)}{\beta^3}.$$

$g^2 \simeq 1.1$ GeV $\qquad b = La \simeq 5 \div 15$ Fm

β	a (Fm)	a^{-1} (GeV)
4.24	0.168	1.17
7.09	0.094	2.09
10.21	0.063	3.12

$N_{meas} \simeq 1000 \qquad n_{copy} \simeq 160 \div 280$

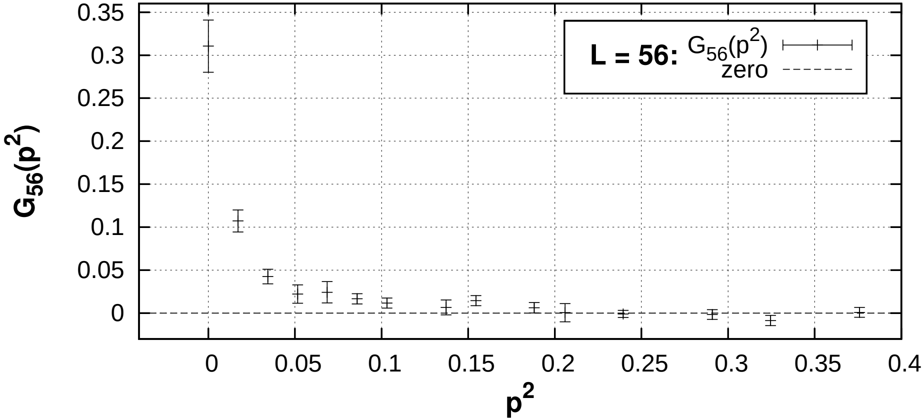

β=10.21
L=36
L=56
L=96
D(p) Gev^-2
|p| Gev
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

Gribov-Stingl fit:

$$D(p) \simeq C \; \frac{(p^2)^v + (d^2)^v}{(p^2 + M^2)^2 + m^4} \tag{21}$$

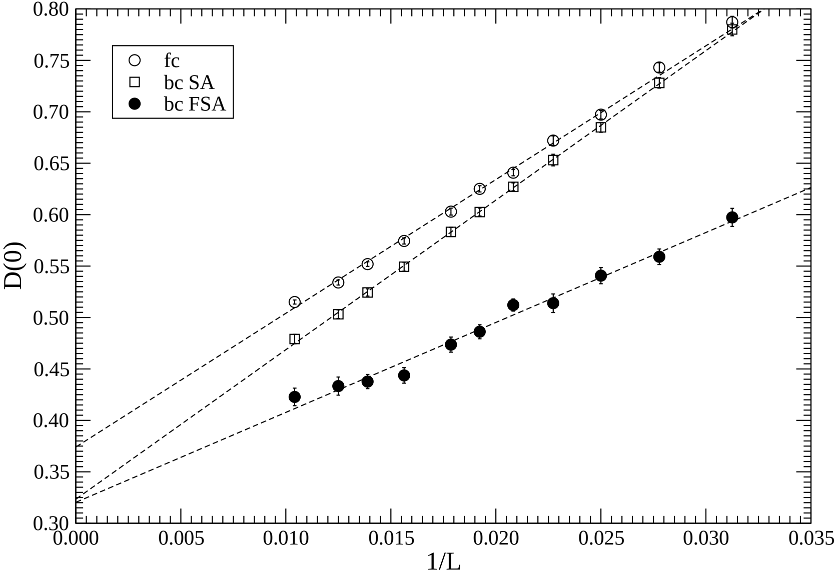
Cucchieri-Dudal-Mendes-Vandersickel fit:

$$D(p) \simeq c_1 \; \frac{(p^2 + c_2^2)(p^2 + c_5^2)}{\left[(p^2 + c_3^2)^2 + c_4^4\right](p^2 + c_6^2)} \tag{22}$$



$$G_L(p) = \frac{D_L^{(first)}(p) - D_L^{(best)}(p)}{D_L^{(best)}(p)} \tag{23}$$

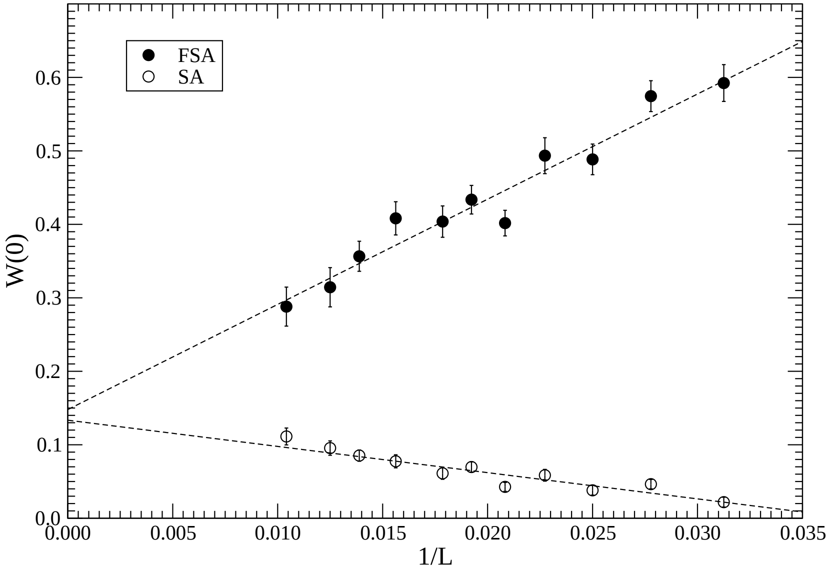
The effect of Gribov copies

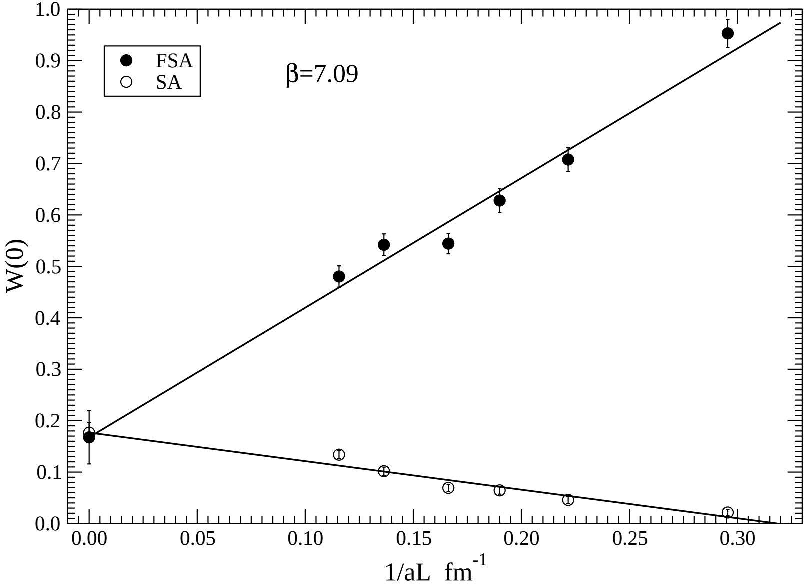


fc
bc SA
bc FSA
D(0)
1/L
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035

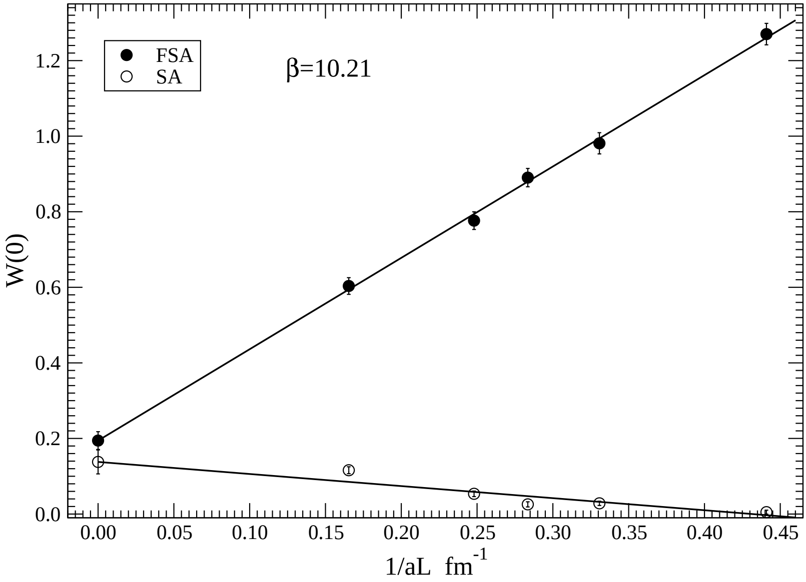
$$G_L(p) = \frac{D_L^{(first)}(p) - D_L^{(best)}(p)}{D_L^{(best)}(p)} \tag{24}$$

$$W_L(0) = \frac{D_L^{(first)}(0) - D_L^{(best)}(0)}{D_\infty^{(best)}(0)} \tag{25}$$

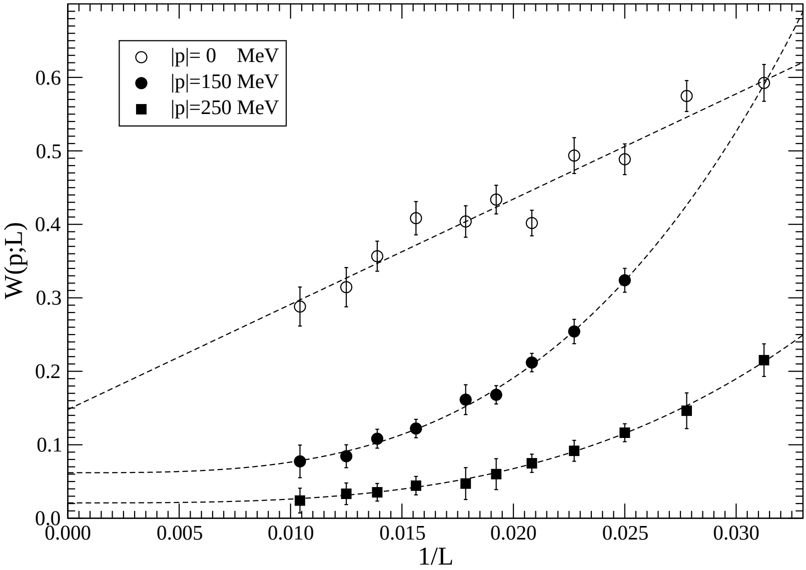

FSA
SA
W(0)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
1/L



FSA
SA
β=7.09
W(0)
1/aL fm-1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.00
0.05
0.10
0.15
0.20
0.25
0.30



FSA
SA
β=10.21
W(0)
1/aL fm-1
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45



|p|= 0 MeV
|p|=150 MeV
|p|=250 MeV
W(p;L)
1/L
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.000
0.005
0.010
0.015
0.020
0.025
0.030

CONCLUSIONS

1. Gribov copy effects survive both thermodynamical limit and continuum limit. This contradicts to the Zwanziger's statement that, in the thermodynamical limit, integrals over the first Gribov region and over FMR are equal to each other. $\Omega_0 \neq \Gamma$!
2. $D(0) \neq 0$. This result is obtained both in the infinite-volume and continuum limits. Thus the scaling solution is ruled out.