# Elastic diffractive scattering of nucleons at ultra-high energies

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**Abstract**

A simple Regge-eikonal model with the eikonal represented as a single-reggeon-exchange term is applied to description of the nucleon-nucleon elastic diffractive scattering at ultra-high energies. The range of validity of the proposed approximation is discussed. The model predictions for the proton-proton cross-sections at the collision energy 14 TeV are given.

## Introduction

The fraction of the elastic diffractive scattering events in the total number of the *pp* collision events at the collision energy 7-8 TeV is about 25% [1]. At higher energies it is expected to be even higher. Hence, understanding of the physical pattern of the small-angle elastic scattering of hadrons is indispensable for general understanding of the strong interaction at ultra-high energies. However, the special status of diffractive studies at high-energy colliders is determined by the fact that diffraction of hadrons takes place due to interaction at large distances. Indeed, the transverse size of the hadron interaction region can be estimated through straight application of the corresponding Heisenberg uncertainty relation to the experimental elastic angular distributions. For example, at the SPS, Tevatron, and LHC energies it is of order 1 fm. Therefore, exploitation of perturbative QCD for treatment of hadronic diffraction is disabled.

The absence of exact theory leads to the emergence of numerous phenomenological models with very different underlying physics (the references to various models of the nucleon-nucleon elastic diffraction can be found in mini-review [2]). The co-existence of a large number of rather complicated (and, often, incompatible) models points to the relevance of the question if construction of a much simpler (but adequate) approximation is possible. The word "adequate" in the last sentence implies as the theoretical correctness, so the satisfactory description of the high-energy evolution of the diffractive pattern.

The aim of this work is to demonstrate that a very simple and physically transparent description can be provided in the framework of the well-known Regge-eikonal approach [3] which originates from the synthesis of Regge theory and quasi-potential approximation (for derivation of the eikonal Regge approximation, see Appendix; for more detailes, see [3] or, for instance, [4]).

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# A model for high-energy elastic diffraction of nucleons

The Regge-eikonal approach to description of the elastic scattering of hadrons exploits the eikonal representation of the non-flip scattering amplitude:

$$T_{el}(s,t) = 4\pi s \int_0^\infty db^2 J_0(b\sqrt{-t}) \frac{e^{2i\delta(s,b)}-1}{2i}\,, \tag{1}$$

$$\delta(s,b) = \frac{1}{16\pi s}\int_0^\infty d(-t) J_0(b\sqrt{-t})\delta(s,t)\,,$$

where $s$ and $t$ are the Mandelstam variables, $b$ is the impact parameter, and eikonal $\delta(s,t)$ is the sum of single-reggeon-exchange terms (see (A.15) in Appendix).

An important advantage of this approach is that it allows to satisfy the Froissart-Martin bound [5] explicitly.

Let us consider the ultimate case of the nucleon-nucleon diffraction at ultra-high energies, where the eikonal can be approximated by the only Regge pole term corresponding to the leading even and $C$-even reggeon (called "pomeron" or "soft pomeron" in literature):

$$\delta(s,t) = \delta_{\rm P}(s,t) \equiv \left(i + {\rm tg}\frac{\pi(\alpha_{\rm P}(t)-1)}{2}\right)\Gamma_{\rm P}{}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_{\rm P}(t)}, \tag{2}$$

where $\alpha_{\rm P}(t)$ is the Regge trajectory of pomeron and $\Gamma_{\rm P}(t)$ is the pomeron form-factor of nucleon.

In the current stage of development, QCD provides no useful information about the behavior of $\alpha_{\rm P}(t)$ and $\Gamma_{\rm P}(t)$ in the diffraction domain ($0 < -t < 2$ GeV$^2$), though it was argued [6] that $\alpha_{\rm P}(t) > 1$ at $t < 0$ and

$$\lim_{t\to-\infty}\alpha_{\rm P}(t) = 1\,. \tag{3}$$

Therefore, we will use the simplest test parametrizations[1]

$$\alpha_{\rm P}(t) = 1 + \frac{\alpha_{\rm P}(0)-1}{1-\frac{t}{\tau_a}}\,,\quad \Gamma_{\rm P}(t) = \frac{\Gamma_{\rm P}(0)}{\left(1-\frac{t}{\tau_g}\right)^2}\,. \tag{4}$$

| **Parameter** | **Value** |
|---|---|
| $\alpha_{\rm P}(0)-1$ | $0.111 \pm 0.017$ |
| $\tau_a$ | $(0.47 \pm 0.12)$ GeV$^2$ |
| $\Gamma_{\rm P}(0)$ | $7.43 \pm 0.94$ |
| $\tau_g$ | $(0.98 \pm 0.12)$ GeV$^2$ |

Table 1: The parameter values for expressions (4), obtained via fitting to the differential cross-section data.

To obtain the angular distribution, one should substitute (4) into (2), then, using representation (1), calculate the scattering amplitude, and, at last, substitute it into the expression for differential cross-section:

$$\frac{d\sigma_{el}}{dt} = \frac{|T_{el}(s,t)|^2}{16\pi s^2}\,. \tag{5}$$

[1] One should not consider the analytic properties of parametrizations (4) seriously. True Regge trajectories and reggeon form-factors have much more complicated analytic structure. Nonetheless, at *negative* values of the argument, they can be approximated by simple monotonic test functions.

To fit the model parameters, we restrict ourselves by the SPS, Tevatron, and LHC energies ($\sqrt{s} > 500$ GeV) and small transfers of momentum ($0.005\ \text{GeV}^2 < -t < 2\ \text{GeV}^2$), since the test parametrization (4) of form-factor $\Gamma_{\mathrm{P}}(t)$ is too stiff and does not allow to provide a satisfactory description of the data in both the diffraction domain and the hard scattering region simultaneously, while at the ISR energies the influence of secondary reggeons becomes significant enough to distort the diffractive pattern crucially (the restriction in the collision energy from below is discussed in detail in the following section).

For verification of the model, we used the set of data on angular distributions collected by J.R. Cudell, A. Lengyel, and E. Martynov at [7]. The original data can be found in [1, 8, 9]. The results of the model application to description of the nucleon-nucleon elastic diffraction are presented in Fig. 1 and Tabs. 1, 2, and 3. As well, in Fig. 1 and Tab. 3 there can be found the predictions for the $pp$ differential, elastic and total cross-sections at various values of the collision energy, including $\sqrt{s} = 14$ TeV.

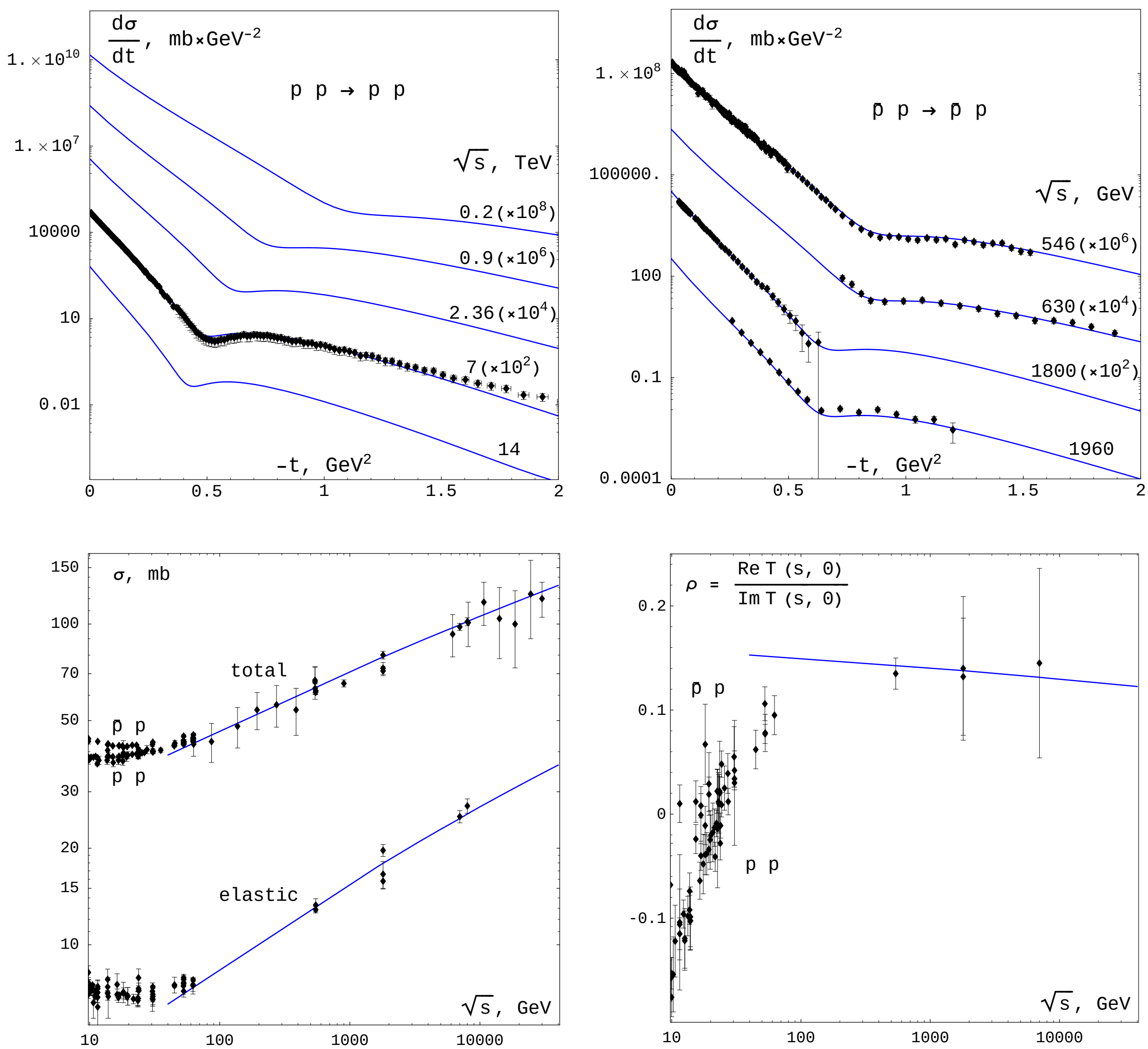


Figure 1: Description of the nucleon-nucleon elastic diffraction observables at ultra-high values of the collision energy.

| $\sqrt{s}$, GeV | **Number of points** | $\chi^2$ |
|---|---|---|
| 546 ($\bar{p}\,p$) | 228 | 222 |
| 630 ($\bar{p}\,p$) | 17 | 13 |
| 1800 ($\bar{p}\,p$) | 51 | 17 |
| 1960 ($\bar{p}\,p$) | 17 | 52 |
| 7000 ($p\,p$) | 161 | 151 |
| **Total** | 474 | 455 |

Table 2: The quality of description of the data on the nucleon-nucleon angular distributions.

| $\sqrt{s}$, GeV | $\sigma_{tot}$, mb | $\sigma_{el}$, mb |
|---|---|---|
| 62.5 | $42.6 \pm 4.0$ | $7.4 \pm 1.1$ |
| 200 | $53.0 \pm 3.5$ | $10.2 \pm 1.0$ |
| 546 | $63.8 \pm 3.3$ | $13.3 \pm 0.9$ |
| 1800 | $79.0 \pm 4.2$ | $18.0 \pm 1.1$ |
| 7000 | $99.6 \pm 7.3$ | $24.8 \pm 2.2$ |
| 8000 | $101.8 \pm 7.7$ | $25.6 \pm 2.3$ |
| 13000 | $110.3 \pm 9.4$ | $28.5 \pm 2.9$ |
| 14000 | $111.7 \pm 9.6$ | $29.0 \pm 3.0$ |

Table 3: Predictions for the *pp* total and elastic cross-sections.

The model description of the nucleon-nucleon diffractive pattern evolution from 0.5 TeV to 7 TeV of the collision energy is satisfactory[2]: $\chi^2/DoF \approx 0.97$.

# The impact of secondary reggeons at the ISR energies

Let us turn to investigation of the proposed approximation validity (or invalidity) at lower energies. The model straight application to description of the *pp* elastic scattering at $\sqrt{s} =$ 62.5 GeV [10] reveals a huge discrepancy between the theoretical curve and the data (Fig. 2, the dashed line), particularly in the dip region. The fact of such a disagreement is not surprising, since the impact of secondary reggeon exchanges is expected to be significant at the ISR energies. Adding some test exponential term to the eikonal, $\delta(s,t) = \delta_{\rm P}(s,t) \to \delta_{\rm P}(s,t) + (i-1)\,\beta\, e^{b\,t}$ (here $s = (62.5\text{ GeV})^2$, $\beta = 6\cdot 62.5^2$, and $b = 4\text{ GeV}^{-2}$), one makes this disagreement much less catastrophic (Fig. 2, the solid line). The dotted line in Fig. 2 corresponds to the replacement $\delta_{\rm P}(s,t) \to \delta_{\rm P}(s,t) + i\,\beta\, e^{b\,t}$ (in the regime ${\rm Re}\,\delta(s,t) \ll {\rm Im}\,\delta(s,t)$, the dip position is determined by ${\rm Im}\,\delta$, while the dip depth is determined by ${\rm Re}\,\delta$).

The question is whether a comparable modification (*i.e.* a few percents increase of ${\rm Im}\,\delta(s,t)$ and a 2-3 times decrease of ${\rm Re}\,\delta(s,t)$) can be caused by the combined contribution of secondary reggeons into the eikonal at the *pp* collision energy 62.5 GeV. Four secondary reggeons are expected to give a noticeable contribution at the ISR energies: two even and $C$-even reggeons, $f$ and $a$, and two odd and $C$-odd reggeons, $\omega$ and $\rho$. With account of these secondaries, the *pp* scattering eikonal takes the following form (see (A.15) in Appendix):

$$ {\rm Im}\,\delta(s,t) = {\rm Im}\,\delta_{\rm P}(s,t) + {\rm Im}\,\delta_{\rm R}(s,t) \equiv {\Gamma_{\rm P}}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_{\rm P}(t)} + $$

[2]At $\sqrt{s} = 546$ GeV, 3 outlying points were excluded from the fitting procedure: at $t_1 = -0.01375\text{ GeV}^2$, $t_2 = -0.034\text{ GeV}^2$, and $t_3 = -1.21\text{ GeV}^2$.

$$+\left[{\Gamma_f}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_f(t)} - {\Gamma_\omega}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_\omega(t)} + {\Gamma_a}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_a(t)} - {\Gamma_\rho}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho(t)}\right],$$

$$\mathrm{Re}\,\delta(s,t) = \mathrm{Re}\,\delta_{\mathrm{P}}(s,t) + \mathrm{Re}\,\delta_{\mathrm{R}}(s,t) \equiv \mathrm{tg}\frac{\pi(\alpha_{\mathrm{P}}(t)-1)}{2}{\Gamma_{\mathrm{P}}}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_{\mathrm{P}}(t)} + \tag{6}$$

$$+\left[\mathrm{tg}\frac{\pi(\alpha_f(t)-1)}{2}\,{\Gamma_f}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_f(t)} + \mathrm{ctg}\frac{\pi(\alpha_\omega(t)-1)}{2}\,{\Gamma_\omega}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_\omega(t)} + \right.$$

$$\left. + \,\mathrm{tg}\frac{\pi(\alpha_a(t)-1)}{2}\,{\Gamma_a}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_a(t)} + \mathrm{ctg}\frac{\pi(\alpha_\rho(t)-1)}{2}\,{\Gamma_\rho}^2(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho(t)}\right].$$

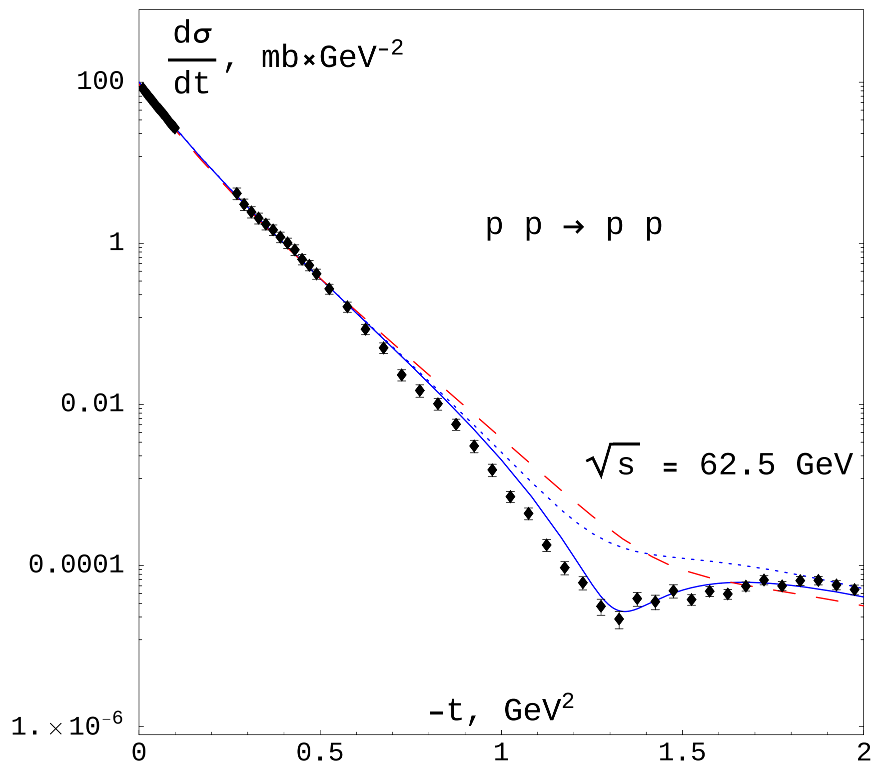


Figure 2: The angular distribution of the $pp$ elastic scattering at $\sqrt{s} = 62.5$ GeV.

The secondary reggeon intercepts can be estimated with the help of their linear Chew-Frautschi plots[3] (Fig. 3): $\alpha_f(0) = 0.68$, $\alpha_\omega(0) = 0.44$, $\alpha_a(0) = \alpha_\rho(0) = 0.47$, and, thus, $\mathrm{tg}\frac{\pi(\alpha_f(0)-1)}{2} \approx -0.55$, $\mathrm{ctg}\frac{\pi(\alpha_\omega(0)-1)}{2} \approx -0.8$, $\mathrm{tg}\frac{\pi(\alpha_a(0)-1)}{2} \approx -1.1$, $\mathrm{ctg}\frac{\pi(\alpha_\rho(0)-1)}{2} \approx -0.9$. Regarding the $t$-evolution of the leading $\bar{q}q$-reggeons, one should keep in mind that $\alpha_{\bar{q}q}(t) \sim \ln^{-1/2}(-t)$ at $t \to -\infty$ [13]. Hence, in the region $t < 0$, the true secondary Regge trajectories should be essentially nonlinear and very different from the corresponding Chew-Frautschi plots. Therefore, for all of the mentioned secondaries, we expect the real parts of their signature factors

[3] The resonance masses are determined from the equations like $\alpha(M^2 - \mathrm{i}M\Gamma) = J$ (here $\alpha(t)$ is some Regge trajectory and $J$ is the resonance spin) which do not imply that $\mathrm{Re}\,\alpha(M^2) = J$ and $\mathrm{Im}\,\alpha(M^2) = 0$. The decay widths $\Gamma$ are only few times less than the corresponding masses $M$: for example, for $f_2(1270)$-meson $\frac{\Gamma_f}{M_f} \approx 0.15$, and for $\rho(770)$-meson $\frac{\Gamma_\rho}{M_\rho} \approx 0.19$. Consequently, the Chew-Frautschi plots should be considered just as very rough approximations to the true Regge trajectories in the resonance region.

to be negative at $t < 0$ and to be, at least, several times higher in magnitude than the factor $\mathrm{tg}\frac{\pi(\alpha_{\mathrm{P}}(t)-1)}{2} < \mathrm{tg}\frac{\pi(\alpha_{\mathrm{P}}(0)-1)}{2} \approx 0.2$. As well, it should be noted that in the $pp$ elastic scattering the contributions of $\omega$ and $\rho$ into $\mathrm{Im}\,\delta(s,t)$ partially compensate the contributions of $f$ and $a$, while in $\mathrm{Re}\,\delta(s,t)$ the corresponding terms have the same (negative) sign (see (6)). Under such conditions, it seems quite natural that, at the $pp$ collision energy 62.5 GeV and low values of the transferred momentum, the combined contribution of secondary reggeons makes $\mathrm{Im}\,\delta(s,t)$ a few percents higher than $\mathrm{Im}\,\delta_{\mathrm{P}}(s,t)$ and makes $\mathrm{Re}\,\delta(s,t)$ two-three times lower than $\mathrm{Re}\,\delta_{\mathrm{P}}(s,t)$.

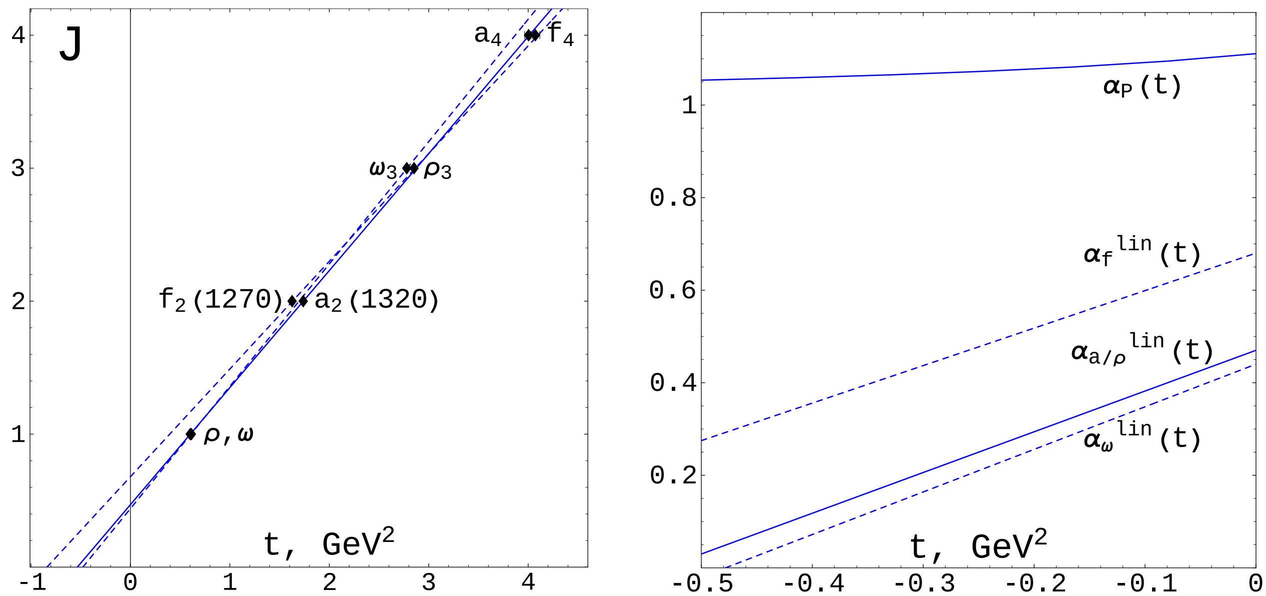


Figure 3: The Chew-Frautschi plots of secondary reggeons in the resonance region (on the left) and in the diffractive scattering region (on the right): $\alpha_f(t) = 0.68 + 0.81t$, $\alpha_\omega(t) = 0.44 + 0.92t$, $\alpha_{a/\rho}(t) = 0.47 + 0.88t$.

In the framework of the discussed physical pattern, it is possible to explain the noticeable discrepancy between the $\bar{p}p$ and $pp$ differential cross-sections in some vicinity of the dip (Fig. 4) without any appeal to the notion of odderon (the leading $C$-odd reggeon with the intercept higher than 1.0). Due to the suppression of the pomeron term in $\mathrm{Re}\,\delta(s,t)$ by the factor $\mathrm{tg}\frac{\pi(\alpha_{\mathrm{P}}(t)-1)}{2}$, the relative contribution of $\omega$ and $\rho$ into $\mathrm{Re}\,\delta(s,t)$ is much more significant than into $\mathrm{Im}\,\delta(s,t)$. These real terms are positive for $\bar{p}p$ scattering and negative for $pp$ scattering. Therefore, $\mathrm{Re}\,\delta(s,t)$ for $\bar{p}p$ scattering could be noticeably higher than for $pp$ scattering. As the angular distribution behavior in some small vicinity of the dip depends on $\mathrm{Re}\,\delta$ strongly, so, in this vicinity, the $\bar{p}p$ differential cross-section is expected to be appreciably larger than the $pp$ differential cross-section (and this is the very pattern which takes place in experiment). If the odderon exchange effect is really negligible, the splitting between the $\bar{p}p$ and $pp$ angular distributions should vanish at higher energies.

In view of the above-said, we infer that although the one-reggeon (pomeron) eikonal approximation to the nucleon-nucleon elastic scattering amplitude is invalid at the ISR energies, the deviations from the experimental data on the corresponding differential cross-sections are, in principle, consistent with the expected phenomenology of secondary reggeons at low negative values of $t$.

The influence of secondaries decreases fast with the collision energy growth. For instance, at $\sqrt{s} = 200$ GeV the combined relative contribution of secondary reggeon exchanges is expected to be about 1-2 percents for $\mathrm{Im}\,\delta(s,t)$ and about 10-20 percents for $\mathrm{Re}\,\delta(s,t)$. Therefore, the model predictions of the $pp$ scattering observables at the RHIC energy (see Fig. 1) could be adequate not only for $\sigma_{tot}$ and $\sigma_{el}$, but also for $\frac{d\sigma_{el}}{dt}$, excluding the very vicinity of the dip. In other words, at the collision energy about 200 GeV the nucleon-nucleon elastic diffraction passes into the pure pomeron-exchange regime.

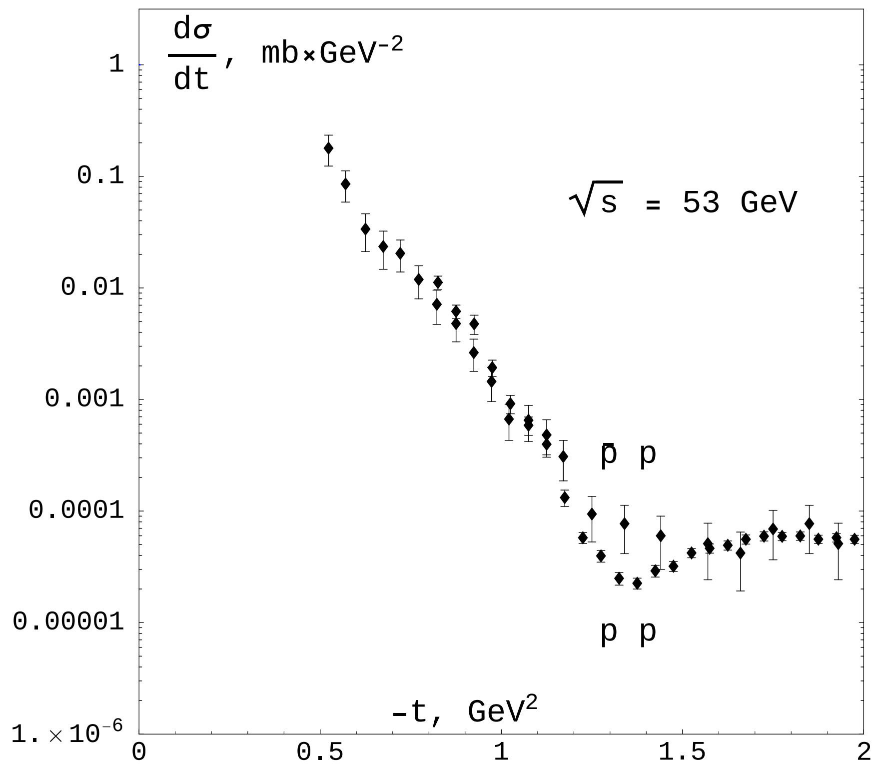


Figure 4: The experimental angular distributions of the $\bar{p}p$ [11] and $pp$ [12] elastic scattering at $\sqrt{s} = 53$ GeV.

# Conclusions

The practical benefit of the one-reggeon eikonal approximation is that, in the kinematic range of diffractive scattering, it allows to reduce the unknown complex function of two variables, $T(s,t)$, to 2 monotonic functions of one variable, $\alpha_{\mathrm{P}}(t)$ and $\Gamma_{\mathrm{P}}(t)$. In spite of its roughness[4], the proposed phenomenological scheme provides a satisfactory description of the nucleon-nucleon elastic scattering at the collision energies 0.54 TeV $\leq \sqrt{s} \leq$ 7 TeV and transferred momenta squared 0.005 GeV$^2$ $< -t <$ 2 GeV$^2$ and allows to give well-grounded predictions for the diffractive pattern at higher (the LHC) and lower (the RHIC) energies.

Nonetheless, confirmation or discrimination of the model is possible after the forthcoming TOTEM measurements of the $pp$ differential cross-section at $\sqrt{s} = 14$ TeV (the analogous measurements at the RHIC are, as well, very desirable). In the case of confirmation, the one-reggeon

[4]Certainly, for better description of experimental data, one could use more complicated and flexible parametrizations of $\alpha_{\mathrm{P}}(t)$ and, especially, of $\Gamma_{\mathrm{P}}(t)$ than stiff test expressions (4).

eikonal approximation could become an effective tool for treatment of the nucleon-nucleon elastic diffraction at ultra-high energies, due to its salient simplicity and physical clearness.

Besides, the pomeron Regge trajectory and the pomeron form-factor of nucleon are used in the framework of the Regge approach to more complicated (than $2 \to 2$) diffractive reactions: single diffraction ($p+p \to p+X$ or $\bar{p}+p \to \bar{p}+X$), central exclusive diffractive production of the Higgs boson ($p+p \to p+H+p$), *etc...* Therefore, the possibility of implicit extraction of $\alpha_{\rm P}(t)$ and $\Gamma_{\rm P}(t)$ from the experimental angular distributions of nucleon-nucleon elastic scattering is very important for raising the predictive efficiency of the reggeon models describing the energy evolution of the corresponding cross-sections.

# Appendix. Derivation of the eikonal Regge approximation.

The eikonal representation (1) itself does not lead to any progress in solving the problem, since it is reduced to the replacement of the unknown function of two variables, $T(s,t)$, to another one, $\delta(s,t)$, without any specification of the functional form of $\delta(s,t)$. The key assumption is that the eikonal is proportional (with high accuracy) to the effective relativistic quasi-potential of hadronic interaction. According to the Van Hove interpretation [14] of the relativistic quasi-potential as the "sum" over all single-particle exchanges in the $t$-channel, the eikonal can be represented in the form

$$\delta^{(f_1,f_2)}(s,t) = \sum_{j=0}^{\infty}\sum_{m_j} J^{(f_1,j,m_j)}_{\alpha_1\dots\alpha_j}(p_1,\Delta)\frac{D^{\alpha_1\dots\alpha_j,\beta_1\dots\beta_j}_{(j,m_j)}(\Delta)}{m_j^2-\Delta^2} J^{(f_2,j,m_j)}_{\beta_1\dots\beta_j}(p_2,\Delta)\,, \tag{A.1}$$

where $\frac{D^{\alpha_1\dots\alpha_j,\beta_1\dots\beta_j}_{(j,m_j)}}{m_j^2-\Delta^2}$ is the propagator of the spin-$j$ particle with mass $m_j$, $J^{(f,j,m_j)}_{\alpha_1\dots\alpha_j}$ is the hadronic current (index $f$ denotes the sort of scattering particle), $\Delta$ is the transferred 4-momentum, $t=\Delta^2$, $p_1$ and $p_2$ are the 4-momenta of the incoming hadrons, $s=(p_1+p_2)^2$, symbol $\sum_{m_j}$ denotes the summing over all of spin-$j$ particles with different masses (which, in what further, will be transformed into the summing over reggeons). We impose the following constraints on the general $\Delta$-dependence of hadronic currents: symmetry with respect to all $\alpha_k$, transversality with respect to $\Delta_{\alpha_k}$ ($k=1,...,j$), and tracelessness with respect to any pair of Lorentz indices. The first two conditions yield

$$J^{(f,j,m_j)}_{\alpha_1\dots\alpha_j}(p,\Delta) = \sum_{k=0}^{\left[\frac{j}{2}\right]}\Gamma^{(f,j,m_j)}_k(p^2,\Delta^2,(p\Delta))\sum G_{\alpha_{\mu_1}\alpha_{\mu_2}}...G_{\alpha_{\mu_{2k-1}}\alpha_{\mu_{2k}}}P_{\alpha_{\mu_{k+1}}}...P_{\alpha_{\mu_j}}\,, \tag{A.2}$$

where $\Gamma^{(f,j,m_j)}_k(p^2,\Delta^2,(p\Delta))$ are some scalar functions, $P_\alpha \equiv \frac{p_\alpha-\frac{p\Delta}{\Delta^2}\Delta_\alpha}{\sqrt{p^2-\frac{(p\Delta)^2}{\Delta^2}}}$, $G_{\alpha\beta}\equiv -g_{\alpha\beta}+\frac{\Delta_\alpha\Delta_\beta}{\Delta^2}$, and the inner sum is over all the nonequivalent permutations of Lorentz indices (the total number of terms is $\frac{j!}{(2k)!!(j-2k)!}$). In view of that

$$g^{\alpha\beta}G_{\alpha\beta}=-3\,,\quad P_\alpha P^\alpha = 1\,,\quad g^{\alpha\beta}G_{\alpha\gamma}G_{\beta\delta}=-G_{\gamma\delta}\,,\quad G_{\alpha\beta}P^\beta=-P_\alpha\,, \tag{A.3}$$

the tracelessness condition results in the recurrent relations

$$\Gamma^{(f,j,m_j)}_k(p^2,\Delta^2,(p\Delta)) = \frac{\Gamma^{(f,j,m_j)}_{k-1}(p^2,\Delta^2,(p\Delta))}{2(j-k)+1}\,, \tag{A.4}$$

and, consequently, yields

$$J^{(f,j,m_j)}_{\alpha_1...\alpha_j}(p,\Delta) = \Gamma_0^{(f,j,m_j)}(p^2,\Delta^2,(p\Delta))\times \tag{A.5}$$

$$\times\sum_{k=0}^{[\frac{j}{2}]}\frac{(2(j-k)-1)!!}{(2j-1)!!}\sum G_{\alpha_{\mu_1}\alpha_{\mu_2}}...G_{\alpha_{\mu_{2k-1}}\alpha_{\mu_{2k}}}P_{\alpha_{\mu_{k+1}}}...P_{\alpha_{\mu_j}}\,.$$

Substituting (A.5) into (A.1) and taking account of the transversality and tracelessness of hadronic currents, we come to the following expression for the eikonal:

$$\delta^{(f_1,f_2)}(s,t) = \sum_{j=0}^{\infty}\sum_{m_j}\frac{\gamma^{(f_1,f_2,j,m_j)}(\Delta^2)}{m_j^2-\Delta^2}\frac{2^j(j!)^2}{(2j)!}P_j\left(\frac{p_1p_2-\frac{(p_1\Delta)(p_2\Delta)}{\Delta^2}}{\sqrt{(p_1^2-\frac{(p_1\Delta)^2}{\Delta^2})(p_2^2-\frac{(p_2\Delta)^2}{\Delta^2})}}\right), \tag{A.6}$$

where $P_j(x)$ are the Legendre polynomials of power $j$ and

$$\gamma^{(f_1,f_2,j,m_j)}(\Delta^2) \equiv \Gamma_0^{(f_1,j,m_j)}(p_1^2,\Delta^2,(p_1\Delta))\Gamma_0^{(f_2,j,m_j)}(p_2^2,\Delta^2,(p_2\Delta))\,. \tag{A.7}$$

For elastic scattering we have

$$(p_1-\Delta)^2 = p_1^2 = m_1^2\,,\quad (p_2+\Delta)^2 = p_2^2 = m_2^2\,, \tag{A.8}$$

and, using the kinematic relations

$$p_{1,2}\Delta = \pm\frac{\Delta^2}{2}\,,\quad p_1p_2 = \frac{s-m_1^2-m_2^2}{2}\,,\quad \Delta^2\equiv t\,, \tag{A.9}$$

we can simplify the expression for the eikonal (from now on, the indices $f_1$ and $f_2$ which denote the sorts of colliding particles will be omitted):

$$\delta(s,t) = \sum_{j=0}^{\infty}\sum_{m_j}\frac{\gamma^{(j,m_j)}(t)}{m_j^2-t}\frac{2^j(j!)^2}{(2j)!}P_j\left(\frac{s-m_1^2-m_2^2+\frac{t}{2}}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right). \tag{A.10}$$

Here we divide the sum over $j$ in the right-hand side of (A.10) into two sums over even and odd $j$:

$$\delta(s,t) = \sum_{j=0}^{\infty}\sum_{\eta=\pm1}\sum_{m_{(\eta)j}}\frac{\eta+e^{-i\pi j}}{2}(-1)^j\frac{\gamma^{(\eta,j,m_{(\eta)j})}(t)}{m^2_{(\eta)j}-t}\frac{2^j(j!)^2}{(2j)!}P_j\left(\frac{s-m_1^2-m_2^2+\frac{t}{2}}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right). \tag{A.11}$$

Each of sequences $\gamma^{(\eta,j,m_{(\eta)j})}(t)$, $m^2_{(\eta)j}$ ($j=0,2,4,...,2n,...$ at $\eta=+1$ and $j=1,3,4,...,2n-1,...$ at $\eta=-1$) separately satisfies the conditions of the Carlson theorem [3, 15] which states that if for some analytic and regular at $\mathrm{Re}\,x\geq 0$ function $f(x)$ the condition $f(x)=O(e^{k|x|})$ with $k<\pi$ is fulfilled, then this function is determined unilocally by its values at integer $x$.

Now we make a key assumption about the possibility of the unilocal (under the Carlson theorem) analytic continuation of (A.11) into the region of complex $j$ (the Regge hypothesis [3]). It implies that $m^2_{(\eta)j}$ and $\gamma^{(\eta,j,m_{(\eta)j})}(t)$ in (A.11) are the values of some analytic (holomorphic with respect to complex variable $j$) functions for integer non-negative values of $j$. We denote these functions by $m^2_\eta(j)$ and $\gamma^{(\eta)}(t,j,m^2_\eta(j))$, respectively. Via the Sommerfeld-Watson transform [3, 16], we replace the sum over $j$ in (A.11) by the integral over the contour $C$ (on the complex plane of variable $j$) encircling the real positive half-axis, including the point $j=0$, in such a way that the half-axis is on the right:

$$\delta(s,t) = \frac{1}{2i}\oint_C\frac{dj}{\sin(\pi j)}\sum_{\eta=\pm1}\sum_{m_\eta}\left(-\frac{\eta+e^{-i\pi j}}{2}\right)\frac{\gamma^{(\eta)}(t,j,m^2_\eta(j))}{m^2_\eta(j)-t}\times \tag{A.12}$$

$$\times \frac{2^j \Gamma^2(j+1)}{\Gamma(2j+1)} P\left(j\ ,\ \frac{s-m_1^2-m_2^2+\frac{t}{2}}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right)$$

(here $P(j,x)$ is the analytic continuation of Legendre polynomials $P_j(x)$ into the region of complex $j$, and $\Gamma(j)$ is the Euler gamma-function). Since, according to our assumption, the unique sources of singularities of the integrand in the region Re $j > -\frac{1}{2}$ are the zeros of functions $\sin(\pi j)$ and $m_\eta^2(j) - t$, then, deforming the contour $C$ and passing to the contour parallel to the imaginary axis, Re $j = -\frac{1}{2}$, we obtain

$$\delta(s,t) = \frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dj}{\sin(\pi j)} \sum_{\eta=\pm 1}\sum_{m_\eta}\left(-\frac{\eta+e^{-i\pi j}}{2}\right)\frac{\gamma^{(\eta)}(t,j,m_\eta^2(j))}{m_\eta^2(j)-t}\times$$

$$\times \frac{2^j \Gamma^2(j+1)}{\Gamma(2j+1)} P\left(j\ ,\ \frac{s-m_1^2-m_2^2+\frac{t}{2}}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right) + \qquad \text{(A.13)}$$

$$+\sum_{\eta=\pm 1}\sum_n\left(-\frac{\eta+e^{-i\pi\alpha_n^{(\eta)}(t)}}{\sin(\pi\alpha_n^{(\eta)}(t))}\right)\frac{d\alpha_n^{(\eta)}(t)}{dt}\,\frac{\pi\gamma^{(\eta)}(t,\alpha_n^{(\eta)}(t),t)}{2}\times$$

$$\times \frac{2^{\alpha_n^{(\eta)}(t)}\Gamma^2(\alpha_n^{(\eta)}(t)+1)}{\Gamma(2\alpha_n^{(\eta)}(t)+1)} P\left(\alpha_n^{(\eta)}(t)\ ,\ \frac{s-m_1^2-m_2^2+\frac{t}{2}}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right)\ ,$$

where functions $\alpha_n^{(\eta)}(t)$ are the roots of the equations $m_\eta^2(j) - t = 0$ and, thus, they correspond to the poles of the eikonal in the region of complex $j$. These poles are called Regge poles, and functions $\alpha_n^{(\eta)}(t)$ are called Regge trajectories (even at $\eta = +1$ and odd at $\eta = -1$).

At $s \gg m_1^2 + m_2^2 - \frac{t}{2}$ we can neglect the contribution of the background integral, and the Legendre polynomials can be described by the leading terms of the expansion. Therefore, it is convenient to introduce new functions

$$\beta_n^{(\eta,s_0)}(t) \equiv \frac{d\alpha_n^{(\eta)}(t)}{dt}\,\frac{\pi\gamma^{(\eta)}(t,\alpha_n^{(\eta)}(t),t)}{2}\left(\frac{s_0}{2\sqrt{(m_1^2-\frac{t}{4})(m_2^2-\frac{t}{4})}}\right)^{\alpha_n^{(\eta)}(t)}\ , \qquad \text{(A.14)}$$

where $s_0$ is any scale determined *a priori* (for example, $s_0 = 1$ GeV$^2$). Note, that functions $\beta_n^{(\eta,s_0)}(t)$ depend on the chosen scale $s_0$ and can be factorized into two factors (see A.7) corresponding to the reggeon form-factors of two interacting hadrons. In the limit of high energies we obtain

$$\delta(s,t) = \sum_n \xi(\alpha_n^+(t))\Gamma_n^{(1)+}(t)\Gamma_n^{(2)+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^+(t)} \mp \sum_n \xi(\alpha_n^-(t))\Gamma_n^{(1)-}(t)\Gamma_n^{(2)-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-(t)}\ , \qquad \text{(A.15)}$$

where $\Gamma_n^{(i)}(t)$ are the reggeon form-factors of the colliding particles, $\xi(\alpha(t))$ are the so-called reggeon signature factors, $\xi(\alpha(t)) = i + \mathrm{tg}\frac{\pi(\alpha(t)-1)}{2}$ for even reggeons, $\xi(\alpha(t)) = i - \mathrm{ctg}\frac{\pi(\alpha(t)-1)}{2}$ for odd reggeons, and the sign "−" ("+") before the $C$-odd reggeon contributions corresponds to the particle-particle (particle-antiparticle) interaction.

The last formula[5], together with the eikonal representation of the scattering amplitude (1), is the essence of the Regge-eikonal model [3].

[5]It is valid, as well, for those single-particle exchanges where (A.8) is violated: as for the inelastic scattering $2 \to 2$, so for reactions with off-shell particles.

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