



Impact-parameter analysis of TOTEM data at the LHC: Black disk limit exceeded

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The bound on inelastic diffraction at the LHC

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Abstract

In this talk we discuss the high-energy dependencies of diffractive and non-diffractive inelastic cross-sections in view of the recent LHC data which reveal a presence of the reflective scattering mode.

Introduction

The measurements performed by the experiments ALICE, ATLAS, CMS, LHCb and TOTEM at the LHC have confirmed an increase with energy of the total, elastic and inelastic cross-sections, the trend earlier observed at lower energies (cf. [1] for interpretation of the new data and the respective references). The results of these measurements are bringing us closer to a revelation of the elusive asymptotic regime of strong interactions.

The analysis of the data on elastic scattering obtained by the TOTEM at $\sqrt{s} = 7$ TeV has revealed an existence of the new regime in strong interaction dynamics, related to transition to the new scattering mode described in [2, 3, 4, 5], anti-shadowing or reflective scattering at very high energies. Experimentally, its appearance is manifested under a reconstruction of the elastic amplitude, elastic and inelastic overlap functions in the impact parameter representation [6]. The nomenclature of reflective scattering is not a widely used one and should be clarified in what follows.

The main issue of this note is related to discussion of the reflective scattering mode, its influence and manifestation in the inelastic diffraction at the LHC. In particular, an upper bound on the inelastic diffractive cross-section in the case when this mode starts to be observed is obtained.

1 Reflective and absorptive scattering modes

The unitarity equation in the impact parameter representation assumes the two scattering modes, which can be designated as absorptive and reflective ones and the particular selection will be described below. An attractive feature of the impact parameter picture is diagonalisation of the unitarity equation written for the elastic scattering amplitude $f(s, b)$, i.e.

$$\text{Im}f(s, b) = |f(s, b)|^2 + h_{inel}(s, b) \quad (1)$$

at high energies with $\mathcal{O}(1/s)$ precision [8], with b being an impact parameter of the colliding hadrons. The $|f(s, b)|^2$ is the elastic channel contribution $h_{el}(s, b)$, while the inelastic overlap function $h_{inel}(s, b)$ covers the sum of the contributions from all the intermediate inelastic channels. The elastic scattering S -matrix element is related to the elastic scattering amplitude $f(s, b)$ by the equation $S(s, b) = 1 + 2if(s, b)$ and can be represented in the form

$$S(s, b) = \kappa(s, b) \exp[2i\delta(s, b)]$$

with the two real functions $\kappa(s, b)$ and $\delta(s, b)$. The function κ ($0 \leq \kappa \leq 1$) is called an absorption factor: its value $\kappa = 0$ corresponds to a complete absorption

of the initial state. At high energies the real part of the scattering amplitude is small and can therefore be neglected, i.e. this fact allows the substitution $f \rightarrow if$ in what follows. It also means that the function $S(s, b)$ is real, but it does not have a definite sign, i.e. it can be positive or negative.

In fact, the choice of elastic scattering mode, namely, absorptive or reflective one, depends on the sign of the function $S(s, b)$, i.e. on the phase $\delta(s, b)$ [7]. The standard assumption is that $S(s, b) \rightarrow 0$ at the fixed impact parameter b and $s \rightarrow \infty$. This is called a black disk limit, and the elastic scattering is completely absorptive. In this case the function $S(s, b)$ is always non-negative. It also implies the limitation $f(s, b) \leq 1/2$.

There is another option: the function $S(s, b) \rightarrow -1$ at fixed b and $s \rightarrow \infty$, i.e. $\kappa \rightarrow 1$ and $\delta = \pi/2$. This phase can be interpreted as the geometric phase related to the presence of singularity [5, 9].

Thus, the function $S(s, b)$ can be negative in the certain region of s - and b - values (i.e. at $s > s_0$ and $0 \leq b < r(s)$). It happens, in particular, in the Donnachie–Landshoff model (cf. [10] and the references therein) at the LHC energies. But, this model does not preserve unitarity, the value of $|S(s, b)|$ eventually exceeds unity at fixed impact parameter when the collision energy being high enough, violating that way a probability conservation. At the LHC energies the amplitude in this case, exceeds the black disk limit at small impact parameters, but, the amplitude itself still obeys the unitarity limitation (cf. [11]).

The exceeding of the black disk limit is a principal conclusion of the model-independent treatment of the impact parameter dependencies performed in paper [6]. This analysis has demonstrated that $f(s, b)$ is greater than black disk limit $1/2$ at $\sqrt{s} = 7$ TeV, but the relative positive deviation α ($f(s, b) = 1/2[1 + \alpha(s, b)]$) is small at this energy¹.

The limiting case $S(s, b) \rightarrow -1$ at fixed b and $s \rightarrow \infty$ can be interpreted as a pure reflective scattering using analogy with a reflection of the light in optics [5]. The appearance of the reflective scattering can be associated with increasing density of a scatterer with energy. It can be said that this density goes beyond the critical value, corresponding to the black disk limit, and that the scatterer starts to reflect the initial wave in addition to its absorption. The principal point of the reflective scattering mode is that $1/2 < f(s, b) \leq 1$ and $0 > S(s, b) \geq -1$, as allowed by unitarity relation [2, 3]. The selection of absorptive or reflective scattering leads to the different values for the ratio $\sigma_{el}(s)/\sigma_{tot}(s)$ at the asymptotical energies, as it will be discussed in what follows.

Indeed, the arguments based on analyticity and unitarity of the scattering ma-

¹The value of α is about 0.04 at this energy and $b = 0$ [6]. It should be stressed therefore, that the most relevant objects to study starting deviation from the black disk limit are $f(s, b)$ and $h_{el}(s, b)$, but not $h_{inel}(s, b)$ since relative deviation in the latter function is of order α^2 , namely $h_{inel}(s, b) = 1/4[1 - \alpha^2(s, b)]$, where $\alpha(s, b)$ is positive in the region $0 \leq b < r(s)$.

trix have lead to conclusion that the Froissart-Martin bound [12, 13] on the total cross-sections would be saturated at the asymptotic energies [14]. The functional energy dependence of the total cross-sections is often taken to have a $\ln^2 s$ -dependence at very high energies, but the value of the factor in front of $\ln^2 s$ remains to be an issue. The value of this factor is related to the choice of the upper limit for the partial amplitude (or the amplitude in the impact parameter representation). The value of this limit may correspond to the maximum of the inelastic channel contribution to the elastic unitarity, when

$$\sigma_{el}(s)/\sigma_{tot}(s) \rightarrow 1/2, \quad (2)$$

or it might correspond to a maximal value of the partial amplitude allowed by unitarity resulting in the asymptotical limit

$$\sigma_{el}(s)/\sigma_{tot}(s) \rightarrow 1. \quad (3)$$

The first option is to be an equivalent of a supposed absorptive nature of the scattering, while the second option assumes an alternative which was interpreted as a reflective scattering (cf. [5] and the above discussion). Assuming absorptive nature of scattering the original Froissart-Martin bound on the total cross-sections has been improved and an upper bound on the total inelastic cross-section reduced by factor of 1/4 has been derived [15]. For the modern status of the bound on the total cross-section and bound on the inelastic cross-section without unknown constants see the recent papers [16] and [17].

It should be noted that the ratio $\sigma_{el}(s)/\sigma_{tot}(s)$ is standing in front of $\ln^2 s$ in the asymptotical bound on the total cross-section [18]:

$$\sigma_{tot}(s) \leq \frac{4\pi}{t_0} \left(\frac{\sigma_{el}(s)}{\sigma_{tot}(s)} \right) \left[\ln \left(\frac{s}{\sigma_{el}(s)} \right) \right]^2 \left[1 + \left(\frac{\text{Re}F(s, t=0)}{\text{Im}F(s, t=0)} \right)^2 \right]^{-1}. \quad (4)$$

We assumed for simplicity that the scale of s is to be determined by $s_0 = 1 \text{ GeV}^2$, but in fact, this scale is an energy-dependent one and is determined by $\sigma_{el}(s)$ as it is clear from Eq. (4), $\sqrt{t_0}$ is the mass of the lowest state in the t channel² and $F(s, t)$ is the elastic scattering amplitude related to $f(s, b)$ by the Fourier-Bessel transformation.

2 Modified upper bound on the inelastic diffraction

An assumption on absorptive nature of the scattering is a crucial issue for the derivation of the Pumplin bound [19, 20], the upper bound for the cross-section of

²For most cases, $t_0 = 4m_\pi^2$.

the inelastic diffraction³:

$$\sigma_{diff}(s, b) \leq \frac{1}{2}\sigma_{tot}(s, b) - \sigma_{el}(s, b), \quad (5)$$

where

$$\sigma_{diff}(s, b) \equiv \frac{1}{4\pi} \frac{d\sigma_{diff}}{db^2}$$

is the total cross-section of all the inelastic diffractive processes in the impact parameter representation and, respectively,

$$\sigma_{tot}(s, b) \equiv \frac{1}{4\pi} \frac{d\sigma_{tot}}{db^2}, \quad \sigma_{el}(s, b) \equiv \frac{1}{4\pi} \frac{d\sigma_{el}}{db^2}.$$

The Eq. (5) was obtained in the framework of the formalism where the inelastic diffraction is considered to be a result of the different absorption of the relevant states [22, 23]. The respective bound on the non-diffractive cross-section is the following :

$$\sigma_{ndiff}(s, b) \geq \frac{1}{2}\sigma_{tot}(s, b) \quad (6)$$

since $\sigma_{ndiff} = \sigma_{inel} - \sigma_{diff}$. These relations, valid for each value of the impact parameter of the collision, can be integrated over b :

$$\sigma_{diff}(s) \leq \frac{1}{2}\sigma_{tot}(s) - \sigma_{el}(s) \text{ and } \sigma_{ndiff}(s) \geq \frac{1}{2}\sigma_{tot}(s). \quad (7)$$

The experimental status of the Eq. (7) at the LHC energies has been discussed recently in [1] and [24]. It was noted that conclusion on the large magnitude of the inelastic diffraction cross-section follows from comparison of the inelastic cross-section measurements performed by ATLAS [26] and CMS [27] with the TOTEM. In order to reconcile the data of all experiments one needs to assume large value for $\sigma_{diff}(s)$ and essential contribution from the low-mass region. As it was noted in [24], an account for the contribution from this region would lead to a resolution of the inconsistency in the different experimental results noted in [25].

Thus, the data obtained at the LHC demonstrate an approximate energy-independence of the ratio $\sigma_{diff}(s)/\sigma_{inel}(s)$ [28]. At $\sqrt{s} = 7$ TeV this ratio is about 1/3. The ratio $\sigma_{diff}(s)/\sigma_{el}(s)$ is approximately equal to unity and

$$[\sigma_{el}(s) + \sigma_{diff}(s)]/\sigma_{tot}(s) = 0.495^{+0.05}_{-0.06}. \quad (8)$$

The above numbers have been taken from [1].

³The more restrictive, but more complicated bounds have been obtained in [21] under the same assumption on absorptive nature of scattering.

But, in the framework of the absorptive scattering, Eqs. (2) and (7) should be fulfilled simultaneously if the black disk limit is taking place asymptotically, i.e.

$$\sigma_{inel}(s)/\sigma_{tot}(s) \rightarrow 1/2 \quad (9)$$

while

$$\sigma_{diff}(s)/\sigma_{tot}(s) \rightarrow 0 \quad (10)$$

and

$$\sigma_{diff}(s)/\sigma_{inel}(s) \rightarrow 0 \quad (11)$$

at $s \rightarrow \infty$.

The limits Eqs. (9-11) are in contradiction. Indeed, $\sigma_{diff}(s)$ should be, by definition⁴, at least an asymptotically nonvanishing part of the inelastic cross-section $\sigma_{inel}(s)$. In contrast to this definition and the available data, one should conclude from Eq. (11) that the inelastic diffractive processes constitute, in fact, a sub-leading mechanism in the inelastic production and the main role in the increase of $\sigma_{inel}(s)$ is due to the non-diffractive inelastic processes. The above statement is difficult to conform with existing experimental trends observed at the LHC.

There is no such an apparent contradiction in the approach assuming saturation of the unitarity limit as it was discussed above. Indeed, the assumption that unitarity limit is to be saturated asymptotically leads to a slower increase of the inelastic cross-section, i.e. at $s \rightarrow \infty$

$$\sigma_{inel}(s)/\sigma_{tot}(s) \rightarrow 0. \quad (12)$$

It allows one to keep considering the inelastic diffraction as a leading mechanism responsible for the inelastic cross-section growth. In this approach the ratio of the elastic to total cross-section Eq. (3) corresponds to energy increase of the total inelastic cross-section slower than $\ln^2 s$ while both Eqs. (3) and (12) take place. And the available experimental data are consistent with decreasing ratio $\sigma_{inel}(s)/\sigma_{tot}(s)$ when the energy increases.

The model-independent reconstruction of the impact-parameter dependent quantities from the experimental data demonstrates that the black disk limit has been exceeded in the elastic scattering at small values of b [6]. In fact, the elastic scattering S -matrix element $S(s, b) \equiv 1 - 2f(s, b)$, where the elastic amplitude $f(s, b)$ is considered to be a real function, is negative at $0 < b < 0.2$ fm and crosses zero at $b = 0.2$ fm at $\sqrt{s} = 7$ TeV. In particular, this is consistent with the result of the Tevatron data analysis [30].

The possibility of going beyond the black disk limit was discussed in the framework of the rational unitarization and the CDF data obtained at Tevatron

⁴A common approach associates dynamics of the inelastic diffraction processes with one or several Pomeron exchanges. Cf. [24, 29] for discussion.

in [3]. It should be noted that the value of $\text{Im}f(s, b = 0)$ has increased from 0.36 (CERN ISR) to 0.492 ± 0.008 (Tevatron) and it is close to exceeding or saturation the black disk limit in this energy domain[30]. As it was noted [2, 3], exceeding the black disk limit turns the derivation of the Pumplin bound to loose its ground. In fact, this bound is not valid in the range of the small and moderate values of the impact parameter, where the absorptive approach ceases to be applicable.

The Pumplin bound can easily be rewritten in terms of $S(s, b)$ in the form

$$\sigma_{diff}(s, b) \leq \frac{1}{4}S(s, b)[1 - S(s, b)]. \quad (13)$$

This inequality clearly indicates that this relation cannot be applied in the region where $S(s, b)$ is negative. This region is determined by the interval $0 < b < r(s)$, where $r(s)$ is the solution of the equation $S(s, b) = 0$ ⁵. In the above mentioned impact parameter range the obvious restriction

$$\sigma_{diff}(s, b) \leq \sigma_{inel}(s, b) \quad (14)$$

can only be applied. In case of reflective scattering this obvious restriction is not a completely trivial in view that $\sigma_{inel}(s, b)$ has a peripheral impact parameter dependence. But, at $b \geq r(s)$ the scattering is absorptive and, therefore, the original bound on the inelastic diffractive cross-section should be valid.

However, the integrated over all values of b relation should be modified. Namely, in this case it is to be written in the form

$$\bar{\sigma}_{diff}(s) \leq \frac{1}{2}\bar{\sigma}_{tot}(s) - \bar{\sigma}_{el}(s), \quad (15)$$

where $\bar{\sigma}_i(s)$ are the reduced cross-sections:

$$\bar{\sigma}_i(s) \equiv \sigma_i(s) - 8\pi \int_0^{r(s)} bdb\sigma_i(s, b),$$

and $i \equiv diff, tot, el$, respectively. Combining Eqs. (14) and (15), the following inequalities relevant for the LHC energies, can easily be obtained:

$$\sigma_{diff}(s) \leq \sigma_{inel}(s) - 2\pi \int_{r(s)}^{\infty} bdb[1 - S(s, b)] \quad (16)$$

and

$$\sigma_{ndiff}(s) \geq 2\pi \int_{r(s)}^{\infty} bdb[1 - S(s, b)]. \quad (17)$$

⁵There is no inelastic diffraction at the impact parameter value where the black disk limit is reached since $S(s, b) = 0$.

The function $S(s, b)$ can be reconstructed from the experimental data on $d\sigma/dt$ in elastic pp -scattering. Using TOTEM data at $\sqrt{s} = 7$ TeV and value of $r(s)$ extracted from their analysis 0.2 fm[6], one obtain the magnitude of the upper bound on $\sigma_{diff}(s)$ at this energy equal to 25.6 mb. Positive contribution of reflective scattering to the bound at this energy is about 5%. Extrapolating data to the energy $\sqrt{s} = 13$ TeV one can provide an estimate for the bound on $\sigma_{diff}(s)$ and the reflective scattering contribution to it at the level of 28.2 mb and $(6 - 8)\%$, respectively⁶. Those numbers are not extremely large, but nonvanishing.

3 The model consideration

The unitary model for the $S(s, b)$ can also be used to estimate qualitatively the dependencies of the cross-sections $\sigma_{diff}(s)$ and $\sigma_{ndiff}(s)$. The reflective scattering is a characteristic picture of the model. It is based on the rational form of the unitarization and represents the function $S(s, b)$ in the form:

$$S(s, b) = \frac{1 - U(s, b)}{1 + U(s, b)}, \quad (18)$$

The $U(s, b)$ is the generalized reaction matrix element, which is considered to be an input dynamical quantity and it is taken to be a real function. The form (18) is a one-to-one transform and is easily invertible. The various dynamical models can be used for the function $U(s, b)$. To get the qualitative estimates we use the simplified form of this function which conforms to rising total cross-section and analytical properties over the transferred momentum, i.e.

$$U(s, b) = g(s) \exp(-\mu b), \quad (19)$$

where $g(s) \sim s^\lambda$, λ and μ are the constants. Eq. (19) can also be motivated by the model proposed by Heisenberg in 1952 [31].

Then the following asymptotical dependencies will take place⁷:

$$\sigma_{tot}(s) \sim \ln^2 s, \quad \sigma_{el}(s) \sim \ln^2 s, \quad \sigma_{inel}(s) \sim \ln s \quad \text{and} \quad r(s) \sim \ln s. \quad (20)$$

From Eq. (16) it follows that for the ratio $\sigma_{diff}(s)/\sigma_{inel}(s)$ the inequality takes place

$$\frac{\sigma_{diff}(s)}{\sigma_{inel}(s)} \leq 1 - \frac{2\pi}{\sigma_{inel}(s)} \int_{r(s)}^{\infty} b db [1 - S(s, b)]. \quad (21)$$

⁶The extrapolated value of $r(s)$ at this energy is about 0.3 fm.

⁷The explicit expressions for $r(s)$ and $\sigma_{inel}(s)$ are the following

$$r(s) = \frac{1}{\mu} \ln g(s) \quad \text{and} \quad \sigma_{inel}(s) = \frac{8\pi}{\mu^2} \ln(1 + g(s)).$$

From Eqs. (17) and (20) it follows that $\sigma_{ndiff}(s) \sim \ln s$ and second term in Eq. (21) tends to $1/2$ at $s \rightarrow \infty$. In general, to exclude a subleading role of $\sigma_{diff}(s)$, the factor in front of $\ln s$ in $\sigma_{ndiff}(s)$ should be different from the corresponding factor in $\sigma_{inel}(s)$ and the asymptotical dependence of the inelastic diffractive cross-section would be $\sigma_{diff}(s) \sim \ln s$. Thus, in this approach both parts of $\sigma_{inel}(s)$ would have similar asymptotical energy dependencies, which are proportional to $\ln s$, while the ratio of the inelastic diffractive to elastic cross-sections would decrease asymptotically like $1/\ln s$, i.e. the relation

$$\sigma_{diff}(s)/\sigma_{el}(s) \rightarrow 0 \quad (22)$$

will take place at $s \rightarrow \infty$.

It would be also interesting to speculate further and assume the saturation of the bound Eq. (21). It would mean that an asymptotic equipartition of the inelastic cross-section on diffractive and non-diffractive ones occurs.

Conclusion

Thus, one can say that, at least, there is no inconsistency between saturation of the unitarity limit leading to Eq. (3) and the bound on the inelastic diffractive cross-section in the case of reflective scattering, i.e. the reflective scattering limit and the ratio

$$\sigma_{diff}(s)/\sigma_{inel}(s) \rightarrow const.$$

at $s \rightarrow \infty$ can easily be reconciled. The energy-independent ratio $\sigma_{diff}(s)/\sigma_{inel}(s)$ is also consistent with the commonly accepted definition of the inelastic diffraction as a result of the Pomeron exchanges and account for the recent experimental trends found at the LHC.

Note, if one assumes mechanism resulting in saturation of the black disk limit at the asymptotic energies, this is not the case. The black disk limit saturation is usually motivated by the eikonal models. Those models reduce the range of the possible variation of a partial amplitude by factor of $1/2$. This reduction is not in a direct inconsistency with the LHC data yet. However, the most recent analysis provides the strong indications on possibility of crossing the values associated with the black disk limit by the elastic amplitude [6]. Thus, it seems now difficult to conform the behavior of the inelastic diffraction at the LHC energy range to the assumption on the black disk limit saturation at $s \rightarrow \infty$.

The new LHC experiments at higher energies would be definitely helpful for resolving the asymptotical dynamics of the inelastic diffraction and elastic scattering.

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