Уравнения Лагранжа-Пуанкаре для динамических систем с симметрией

С.Н.Сторчак

Talks consists mostly of

1. Introduction

2. The Lagrange–Poincaré equations for mechanical systems with symmetry from the Poincaré variational principle

3. Equations of relative equilibria in Yang-Mills theory

4. The Lagrange–Poincaré equations for a special mechanical system (the mechanical model of the interaction of Yang-Mills fields with the scalar fields)

Dynamical systems with a symmetry

Infinite-dimensional: Yang-Mills theories,...

The gauge group, an invariant Lagrangian $L = -\frac{1}{4g_0^2} F^a_{\mu\nu} F^{a\mu\nu}$

Finite-dimensional: Mechanical systems with symmetry (on manifolds)

A group acting on a manifold, $L = \frac{1}{2} G_{AB}(Q) \dot{Q}^A \dot{Q}^B - V(Q)$

The general properties: Geometry (the Riemannian manifold, the principal fiber bundle, reduction –"removing" symmetry out of the system)
In Y.-M., in the gauge $A_0 = 0$, $L = \frac{1}{2g_0^2}(\partial_0 A_i^a)^2 - \frac{1}{4g_0^2} F^a_{ij} F^{aij}$, removing the rest gauge symmetry $g(\bar{x})$

A scalar particle on Riemannian compact manifold

$\mathcal{P}$ — a smooth compact Riemannian manifold (without boundary); $\mathcal{G}$ semi-simple compact Lie group. The action of $\mathcal{G}$ is smooth isometric and free: (i.e. $pg = p \rightarrow g = e$).

$\Rightarrow$ $\mathcal{P}$ can be regarded as a total space of the principal fiber bundle $\pi : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{G} = \mathcal{M}$. This means that

$\pi^{-1}(U_\alpha) \sim U_\alpha \times \mathcal{G}$; $p \sim (x, g_\alpha)$ or $p = \varphi_\alpha(x, g_\alpha)$.

If $x \in (U_\alpha) \bigcap (U_\beta)$, then $p = \varphi_\beta(x, g_\beta)$. With the transition functions $\varphi_\beta^{-1}\varphi_\alpha = \varphi_{\beta\alpha}(p) = \varphi_{\beta\alpha}(x)$ $\Rightarrow$ $p = \varphi_\beta(x, \varphi_{\beta\alpha}(x) g_\alpha) = \varphi_\alpha(x, g_\alpha)$.

Рис.: The principal fibre bundle

The submanifold Σ in $\mathcal{P}$ (the gauge surface) is given by the equation $\{\chi^{\alpha}(Q) = 0\}$.

The original coordinates Q^A on $\mathcal{P}$ $\rightarrow$ (Q^{*A}, a^{α}) with $\chi^{\alpha}(Q^{*}) = 0$, for a one-to-one mapping.

Bundle coordinates on $\mathcal{P}$

Eq. $\chi^\alpha\big(F^A(Q, a^{-1}(Q))\big) = 0$ gives $a^\alpha(Q)$; $Q^{*A} = F^A(Q, a^{-1}(Q))$

Transformation $Q^A = F^A(Q^{*A}, a^\alpha)$.

$$\frac{\partial}{\partial Q^B} = F^C_B(F(Q^*, a), a^{-1}) N^A_C(Q^*) \frac{\partial}{\partial Q^{*A}}$$
$$+ F^E_B(F(Q^*, a), a^{-1}) \chi^\mu_E(Q^*) (\Phi^{-1})^\beta_\mu(Q^*) \bar{v}^\alpha_\beta(a) \frac{\partial}{\partial a^\alpha}.$$

$\bar{v}^\alpha_\beta(g_2) = \partial \Phi^\alpha(g_1, g_2)/\partial g_1^\beta|_{g_1=e}$, the Faddeev – Popov matrix:

$$(\Phi)^\beta_\mu(Q) = K^A_\mu(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A}$$

(K_μ are the Killing vector fields for the Riemannian metric G_{AB}),
N^A_C project onto the subspace which is orthogonal to K_μ

$$N^A_C(Q) = \delta^A_C - K^A_\alpha(Q)(\Phi^{-1})^\alpha_\mu(Q) \chi^\mu_C(Q).$$

Рис.: The action of the projection operators N and $P_{\perp}$:
$a = \hat{A} + P_{\perp}dA$; $c = \hat{A} + NdA$ lies on the plane normal to the gauge orbit through $\hat{A}$; $b = \hat{A} + (1 - P_{\perp})dA$; $d = \hat{A} + (1 - N)dA$

The metric $\tilde{G}_{\mathcal{AB}}$ on $\mathcal{P}$

In $(\frac{\partial}{\partial Q^{*A}}, \frac{\partial}{\partial a^{\alpha}})$ basis

$$\begin{pmatrix} G_{CD}(Q^*)(P_\perp)^C_A(P_\perp)^D_B & G_{CD}(Q^*)(P_\perp)^D_A K^C_\mu \bar{u}^\mu_\alpha(a) \\ G_{CD}(Q^*)(P_\perp)^C_A K^D_\nu \bar{u}^\nu_\beta(a) & \gamma_{\mu\nu}(Q^*)\bar{u}^\mu_\alpha(a)\bar{u}^\nu_\beta(a) \end{pmatrix},$$

the metric on the orbit – $\gamma_{\mu\nu} = K^A_\mu G_{AB} K^B_\nu$. The projector $P_\perp$ onto the tangent space to the gauge surface Σ:

$$(P_\perp)^A_B = \delta^A_B - \chi^\alpha_B (\chi\chi^\top)^{-1}{}^\beta_\alpha (\chi^\top)^A_\beta,$$

$$N^A_B N^B_C = N^A_C, \quad (P_\perp)^{\tilde{A}}_B N^C_{\tilde{A}} = (P_\perp)^C_B, \qquad N^{\tilde{A}}_B (P_\perp)^C_{\tilde{A}} = N^C_B.$$

$$(\det \tilde{G}_{\mathcal{AB}}) = \det\Big((P_\perp)^D_A \; G^{\rm H}_{DC} \, (P_\perp)^C_B\Big) \det \gamma_{\alpha\beta} \; (\det \bar{u}^\mu_\nu)^2.$$

The "horizontal metric" $G^{\rm H}_{DC} = \Pi^{\tilde{D}}_D \, \Pi^{\tilde{C}}_C \, G_{\tilde{D}\tilde{C}}$; $\Pi^A_B = \delta^A_B - K^A_\mu \gamma^{\mu\nu} K^D_\nu G_{DB}$ is the projection operator.

Transformation of the Lagrangian

$$\mathcal{L} = \frac{1}{2} G_{AB}(Q)\dot{Q}^A\dot{Q}^B - V(Q),\ V(F(Q,a)) = V(Q).$$

$Q^A \to (Q^{*A}, a^{\alpha})$, $\quad Q^A = F^A(Q^{*B}, a^{\alpha})$, (dependent coo. Q^{*A} on Σ.)

$$\dot{Q}^A(t) = F^A_C\Big(\frac{dQ^{*C}}{dt} + K^C_{\beta}(Q^*)\,\bar{u}^{\beta}_{\alpha}(a)\,\frac{da^{\alpha}}{dt}\Big).$$

Since $(P_{\perp})^C_D\,\frac{dQ^{*D}}{dt} = \frac{dQ^{*C}}{dt}$. $\Rightarrow dQ^{*C}/dt \in T\Sigma$.

$$\mathcal{L} = \frac{1}{2} G_{CD}(Q^*)\Big(\frac{dQ^{*C}}{dt} + K^C_{\mu}\,\bar{u}^{\mu}_{\alpha}(a)\,\frac{da^{\alpha}}{dt}\Big)\Big(\frac{dQ^{*D}}{dt} + K^D_{\nu}\,\bar{u}^{\nu}_{\beta}(a)\,\frac{da^{\beta}}{dt}\Big) - V(Q^*).$$

That is, $\mathcal{L}$ is now given on the total space of $\Sigma \times \mathcal{G} \to \Sigma$ (the locally trivial principal bundle). $P(\mathcal{M}, \mathcal{G})$ is considered as a locally trivil and isomorhic to the trivial bundle over Σ.

Nonholonomic basis (H_A, L_α)

$[H_A, L_\alpha] = 0$; $[L_\alpha, L_\beta] = c^\gamma_{\alpha\beta} L_\gamma$.

$$L_\alpha = v^\mu_\alpha(a)\frac{\partial}{\partial a^\mu}, \qquad H_A = N^E_A(Q^*)\left(\frac{\partial}{\partial Q^{*E}} - \tilde{\mathcal{A}}^\alpha_E L_\alpha\right),$$

$\tilde{\mathcal{A}}^\alpha_E(Q^*, a) = \bar{\rho}^\alpha_\mu(a)\,\mathcal{A}^\mu_E(Q^*)$. The matrix $\bar{\rho}^\alpha_\mu$ is inverse to the matrix $\rho^\beta_\alpha = \bar{u}^\alpha_\mu v^\mu_\beta$ of the adjoint representation of the group $\mathcal{G}$.

$\mathcal{A}^\nu_P = \gamma^{\nu\mu} K^R_\mu G_{RP}$ is the "mechanical" connection on $P(\Sigma, \mathcal{G})$.

$$[H_C, H_D] = (\Lambda^\gamma_C N^P_D - \Lambda^\gamma_D N^P_C) K^S_{\gamma P} H_S - N^E_C N^P_D \tilde{\mathcal{F}}^\alpha_{EP} L_\alpha \equiv \mathbb{C}^A_{CD} H_A + \mathbb{C}^\alpha_{CD} L_\alpha,$$

$\Lambda^\gamma_D = (\Phi^{-1})^\gamma_\mu \chi^\mu_D$. $\quad\tilde{\mathcal{F}}^\alpha_{EP}$ is the curvature of the connection $\tilde{\mathcal{A}}$

$$\tilde{\mathcal{F}}^\alpha_{EP} = \frac{\partial}{\partial Q^{*E}}\tilde{\mathcal{A}}^\alpha_P - \frac{\partial}{\partial Q^{*P}}\tilde{\mathcal{A}}^\alpha_E + c^\alpha_{\nu\sigma}\tilde{\mathcal{A}}^\nu_E \tilde{\mathcal{A}}^\sigma_P,$$

the metric becomes

$$\check{G}_{\mathcal{AB}} = \begin{pmatrix} G^{\mathrm{H}}_{AB} & 0 \\ 0 & \tilde{\gamma}_{\alpha\beta} \end{pmatrix}, \quad \check{G}^{\mathcal{AB}} = \begin{pmatrix} G^{EF} N^A_E N^B_F & 0 \\ 0 & \tilde{\gamma}^{\alpha\beta} \end{pmatrix},$$

$$\tilde{G}(H_A, H_B) \equiv G^{\mathrm{H}}_{AB}(Q^*), \quad \tilde{G}(L_\alpha, L_\beta) \equiv \gamma_{\alpha'\beta'}(Q^*)\, \rho^{\alpha'}_{\alpha}(a)\, \rho^{\beta'}_{\beta}(a).$$

The orthogonality condition: $\check{G}^{\mathcal{AB}} \check{G}_{\mathcal{BC}} = \begin{pmatrix} N^A_C & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}.$

The Lagrangian in the nonholonomic basis

$$\hat{\mathcal{L}} = \frac{1}{2}\, G^{\mathrm{H}}_{CD}\, \omega^C \omega^D + \frac{1}{2} \tilde{\gamma}_{\mu\nu} \omega^\mu \omega^\nu - V,$$

$$\text{where } \omega^{\mathrm{E}} = (\mathrm{P}_\perp)^{\mathrm{E}}_{\mathrm{B}} \frac{\mathrm{d}\mathrm{Q}^{*\mathrm{B}}}{\mathrm{dt}} = \frac{\mathrm{d}\mathrm{Q}^{*\mathrm{E}}}{\mathrm{dt}}, \quad \text{and } \omega^\alpha = \mathrm{u}^\alpha_\sigma \frac{\mathrm{d}\mathrm{a}^\sigma}{\mathrm{dt}} + \omega^{\mathrm{D}} \tilde{\mathcal{A}}^\alpha_{\mathrm{D}}.$$

$$\frac{da^\beta}{dt} = v^\beta_\alpha\, \omega^\alpha - \omega^D v^\beta_\alpha\, \tilde{\mathcal{A}}^\alpha_D.$$

ω^α, which is obtained as a result of the shift of the vertical velocity by means of the mechanical connection $\tilde{\mathcal{A}}^\alpha_D$, and the variation of the variable ω^E have a globally intrinsic sense (Marsden-Ratiu).

The Poincaré variational principle

The variations of paths that are associated with independent vector fields.
(Nonholonomic) coordinate basis on a manifold: $v_i : [v_i, v_j] = c_{ij}^k(q) v_k$.
For a path $q(t)$ on a manifold,

$$\frac{df(q(t))}{dt} = \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} = \sum\nolimits_i v_i(f)\omega^i,$$

($v_i(f)$ is the directional derivative of f), ω^i — the quasi-velocities: linear functions of the velocities $\dot{q}^i$.
A path $q(t)$, its deformation $q(u,t)$

$$\frac{\partial f(q(u,t))}{\partial u} = \sum\nolimits_i v_i(f) w^i(u,t).$$

$w^i(u,t)$ –independent variations, $w^k(u,t_1) = 0$ and $w^k(u,t_2) = 0$.
The variation of the functional $F(q(t))$:

$$\delta F = \frac{dF(q(u,t))}{du}\Big|_{u=0}.$$

Relations between partial derivatives of ω^A and w^A

Our case – vector fields, $\{H_A\}$ and $\{L_\alpha\}$, independendence: $[H_A, L_\alpha] = 0$. The quasi-velocities: ω^A and ω^α

$$\frac{df(Q^*, a)}{dt} = \omega^E H_E(f) + \omega^\alpha L_\alpha(f).$$

$f = Q^{*A}:\ \frac{dQ^{*A}(t)}{dt} = \omega^E(t) H_E^A(Q^*(t)),\ \ H_E^A(Q^*) \equiv H_E(Q^{*A}) = N_E^A(Q^*).$

$Q^{*A}(u,t):\ \ \frac{\partial Q^{*A}(u,t)}{\partial t} = H_E(Q^{*A}(u,t))\,\omega^E(u,t),\ \ \omega^E(u,t) = \frac{dQ^{*E}(u,t)}{dt}.$

$\frac{\partial Q^{*A}(u,t)}{\partial u} = H_E(Q^{*A}(u,t))\, w^E(u,t).$

$(\frac{\partial}{\partial u}\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial u})Q^{*A}(u,t) = 0$ + comm. relations $\Longrightarrow$

$$N_R^A\Big(\frac{\partial \omega^R}{\partial u} - \frac{\partial w^R}{\partial t} + \mathbb{C}_{PE}^R\,\omega^E w^P\Big) = 0.$$

$$\frac{\partial \omega^\beta}{\partial u} = \frac{\partial w^\beta}{\partial t} + c^\beta_{\alpha'\mu}\,\omega^{\alpha'} w^\mu + N_E^C N_P^{C'}\tilde{\mathcal{F}}^\beta_{C'C}\,\omega^E w^P.$$

The Lagrange-Poincaré equations

$$\frac{dS}{du} = \int_{t_1}^{t_2} \Big(\frac{\partial \hat{\mathcal{L}}}{\partial \omega^{C'}} \frac{\partial \omega^{C'}}{\partial u} + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^{\mu'}} \frac{\partial \omega^{\mu'}}{\partial u} + \frac{\partial \hat{\mathcal{L}}}{\partial Q^{*B}} \frac{\partial Q^{*B}}{\partial u} + \frac{\partial \hat{\mathcal{L}}}{\partial a^{\alpha}} \frac{\partial a^{\alpha}}{\partial u} \Big) dt.$$

$\frac{\partial \omega^{C'}}{\partial u} \Rightarrow \frac{\partial w^{C'}}{\partial t}$ through the relations and integrating by parts leads to

$$-\frac{d}{dt}\Big(\frac{\partial \hat{\mathcal{L}}}{\partial \omega^{E}} \Big) + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^{P}} \mathbb{C}^{P}_{CE}\, \omega^{C} + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^{\alpha}} N^{C'}_{E}\, \tilde{\mathcal{F}}^{\alpha}_{C'B}\, \omega^{B} + H_{E}(\hat{\mathcal{L}}) = 0.$$

$$-\frac{d}{dt}\Big(\frac{\partial \hat{\mathcal{L}}}{\partial \omega^{\alpha}} \Big) + \frac{\partial \hat{\mathcal{L}}}{\partial \omega^{\mu}} c^{\mu}_{\nu\alpha}\, \omega^{\nu} + L_{\alpha}(\hat{\mathcal{L}}) = 0.$$

The Lagrange-Poincaré equations in coordinates

$$-N^L_D\Big(\frac{d}{dt}\,\omega^D + {}^{\rm H}\Gamma^D_{AB}\,\omega^A\omega^B + G^{DE}N^B_E\mathcal{F}^{\mu}_{AB}\,p_\mu\,\omega^A +$$
$$+\frac{1}{2}G^{DE}N^S_E(\mathcal{D}_S\gamma^{\kappa\sigma})p_\kappa p_\sigma + G^{DE}N^S_E\frac{\partial}{\partial Q^{*S}}V(Q^*)\Big) = 0.$$

$p_\sigma = \gamma_{\alpha\sigma}\rho^\alpha_\epsilon\omega^\epsilon$, $\mathcal{D}_S\gamma^{\kappa\sigma} = \frac{\partial}{\partial Q^{*S}}\gamma^{\kappa\sigma} + c^\kappa_{\mu\nu}\mathcal{A}^\mu_S\gamma^{\nu\sigma} + c^\sigma_{\mu\nu}\mathcal{A}^\mu_S\gamma^{\nu\kappa}$.

$$\frac{d}{dt}p_\beta + c^\nu_{\epsilon\beta}\gamma^{\epsilon\varphi}p_\varphi p_\nu - c^\nu_{\mu\beta}\mathcal{A}^\mu_C\omega^C p_\nu = 0.$$

In resolved gauge case, the invariant coordinate functions $x^i(Q)$, determined by the equation $Q^{*A}(x) = F^A(Q, a^{-1}(Q))$, $\chi^\alpha(Q^{*A}(x)) = 0$, are the coordinates on the base $\mathcal{M}$ of the principal bundle $P(\mathcal{M},\mathcal{G})$).
The horizontal equation is

$$Q^{*L}_i\Big(\ddot{x}^i + \Gamma^i_{mn}\dot{x}^m\dot{x}^n + h^{il}\mathcal{F}^\mu_{kl}\dot{x}^k p_\mu + \frac{1}{2}h^{ij}\left(\mathcal{D}_j\gamma^{\kappa\sigma}\right)p_\kappa p_\sigma + h^{ij}\frac{\partial}{\partial x^j}V(Q^*(x))\Big) = 0$$

The relative equilibrium

A relative equilibrium is such a motion in the unreduced phase space which does not change the “shape” of a system, so the system may be viewed as a rigid body. The motion for which $\omega^A = 0$ ($\dot{Q}^{*A} = 0$).

$$-N^L_D\Big(\frac{d}{dt}\,\omega^D + {}^{\mathrm{H}}\Gamma^D_{AB}\,\omega^A\omega^B + G^{DE}N^B_E\mathcal{F}^{\mu}_{AB}\,p_\mu\,\omega^A + \\ +\frac{1}{2}G^{DE}N^S_E(\mathcal{D}_S\gamma^{\kappa\sigma})p_\kappa p_\sigma + G^{DE}N^S_E\frac{\partial}{\partial Q^{*S}}V(Q^*)\Big) = 0.$$

$$\frac{d}{dt}p_\beta + c^{\nu}_{\epsilon\beta}\gamma^{\epsilon\varphi}p_\varphi p_\nu - c^{\nu}_{\mu\beta}\mathcal{A}^{\mu}_{C}\omega^C p_\nu = 0.$$

From hor. eq. $\Rightarrow \dot{p} = 0$ (but, p_μ may be constant $\neq 0$).

$$G^{DE}N^L_D N^S_E\Big(\frac{1}{2}(\mathcal{D}_S\gamma^{\kappa\sigma})p_\kappa p_\sigma + \frac{\partial}{\partial Q^{*S}}V(Q^*)\Big) = 0, \\ c^{\nu}_{\epsilon\beta}\gamma^{\epsilon\varphi}p_\varphi p_\nu = 0. \qquad (1)$$

(recall that $p_\mu = \gamma_{\mu\sigma}\rho^{\sigma}_{\epsilon}\omega^{\epsilon}$, $\omega^{\epsilon} \sim \frac{da}{dt}$)

On solution of the relative equilibrium equations

$$c^{\nu}_{\epsilon\beta}\gamma^{\epsilon\varphi}p_{\varphi}p_{\nu} = 0.$$

If p_{μ} is an eigenfunction e_{κ} of

$$(k_{\varphi\nu}\gamma^{\nu\kappa})\mathrm{e}_{\kappa} = \lambda\,\mathrm{e}_{\varphi}, \quad \text{where } \mathrm{k}_{\alpha\beta} = \mathrm{c}^{\nu}_{\mu\alpha}\mathrm{c}^{\mu}_{\nu\beta}.0$$

Then, since $\gamma^{\nu\kappa}\mathrm{e}_{\kappa} = \lambda\,k^{\nu\epsilon}\,\mathrm{e}_{\epsilon}$, we have for eq. $\lambda\,c^{\mu}_{\sigma\nu}k^{\nu\epsilon}\,\mathrm{e}_{\epsilon}\mathrm{e}_{\mu} = 0$.
Using the identity $c^{\mu}_{\sigma\nu}k^{\nu\epsilon} = -c^{\epsilon}_{\sigma\nu}k^{\nu\mu}$, we get at the lhs of eq.

$$c^{\mu}_{\sigma\nu}k^{\nu\epsilon}\,\mathrm{e}_{\epsilon}\mathrm{e}_{\mu} = -c^{\epsilon}_{\sigma\nu}k^{\nu\mu}\mathrm{e}_{\epsilon}\mathrm{e}_{\mu}$$

But this means that $c^{\mu}_{\sigma\nu}k^{\nu\epsilon}\,\mathrm{e}_{\epsilon}\mathrm{e}_{\mu} \equiv 0$. $\Rightarrow$
the eigenfunctions e_{κ} are solutions of the second equation of the system. Substitutiing them into the first equation, we get the equation which determines the value of the variable Q^{*A} at the relative equilibrium of the dynamical system. The solution of the system represents a group orbit which is invariant under dynamics. The projection of this orbit defines the equilibrium of the reduced dynamical system.

Principal bundle coordinates in Yang-Mills

In Yang-Mills theory, the evolution is considered on the functional space of gauge connections. The gauge transformation group acts on this space. The reduced evolution given on the orbit space of its action. The functional restrictions are needed (Singer,Narasimhan,Parker,Soloviev).

Assumptions:

The points of a manifold $\mathcal{P}$ are the irreducible connections in the principal fiber bundle $P(M,G)$ (in Sobolev class H_k, $k>3$).

The transformation group $\mathcal{G}$ is the quotient group of the gauge transformation group by its center.

we assume that this group is the gauge group of time independent transformations:[1]

$$\tilde{A}^{\alpha}_i(\mathbf{x}) = \rho^{\alpha}_{\beta}(g^{-1}(\mathbf{x}))A^{\beta}_i(\mathbf{x}) + u^{\alpha}_{\mu}(g(\mathbf{x}))\frac{\partial g^{\mu}(\mathbf{x})}{\partial \mathbf{x}^i}\,,$$

...principal bundle coordinates in Yang-Mills

The Hamiltonian of the pure Yang-Mills field in the Schrödinger functional approach

$$H = \frac{1}{2}\mu^2\kappa\,\triangle_{\mathcal{P}}[A_a] + \frac{1}{\mu^2\kappa}\,V[A_a],$$

$$\triangle_{\mathcal{P}}[A] = \int d^3x\,k^{\alpha\beta}\delta_{ij}\frac{\delta^2}{\delta A^{\alpha}_i(\mathbf{x})\,\delta A^{\beta}_j(\mathbf{x})}\,,$$

$$V[A] = \int d^3x\,\frac{1}{2}\,k_{\alpha\beta}\,F^{\alpha}_{ij}(\mathbf{x})\,F^{\beta\;ij}(\mathbf{x})\,.$$

$k_{\alpha\beta} = c^{\tau}_{\mu\alpha}c^{\mu}_{\tau\beta}$ is the Cartan–Killing metric for a group G, $\mu^2 = \hbar g_0^2$ and κ is a real positive parameter. It follows that in this problem, for the quadratic part of the Hamiltonian,

$$G^{(\alpha,i,x)\;(\beta,j,x')}\frac{\delta^2}{\delta A^{(\alpha,i,x)}\;\delta A^{(\beta,j,x')}},$$

one can use the plane metric $G^{(\alpha,i,x)\;(\beta,j,x')} = \delta^{\alpha\,\beta}\,\delta^{i\,j}\,\delta^3(\mathbf{x}-\mathbf{x}')$.

...principal bundle coordinates in Yang-Mills

A plane Riemannian metric on the original manifold $\mathcal{P}$ of the gauge potentials:

$$ds^2 = G_{(\alpha,i,x)(\beta,j,y)}\delta A^{(\alpha,i,x)}\delta A^{(\beta,j,y)}\,,$$

$$G_{(\alpha,i,x)(\beta,j,y)} = G\left(\frac{\delta}{\delta A_i^{\alpha}(\mathbf{x})}\,,\,\frac{\delta}{\delta A_j^{\beta}(\mathbf{y})}\right) = k_{\alpha\beta}\delta^{ij}\delta^3(\mathbf{x}-\mathbf{y})$$

the principal fiber bundle coordinates $A_i^{\alpha}(\mathbf{x}) \to (A^{*\alpha}_{\;i}(\mathbf{x}), g^{\mu}(\mathbf{x}))$, dependent coordinates A^*: $\chi^{\alpha}(A^*)=0$, χ^{ν} — the Coulomb gauge: $\partial^k A_k^{\nu}(\mathbf{x})=0$.

The Killing vectors $K_{(\alpha,y)}$, $K_{(\alpha,y)} = K^{(\mu,i,x)}_{\quad\;(\alpha,y)}\frac{\delta}{\delta A^{(\mu,i,x)}}\,,$

$$K^{(\mu,i,x)}_{\quad\;(\alpha,y)}(A) \equiv \left[\mathcal{D}^{\mu i}_{\;\;\alpha}(A(\mathbf{x}))\,\delta^3(\mathbf{x}-\mathbf{y})\right]$$

The orbit metric $\gamma_{(\mu,x)(\nu,y)} = K^{(\alpha,i,z)}_{\quad\;(\mu,x)}G_{(\alpha,i,z)(\beta,j,u)}K^{(\beta,j,u)}_{\quad\;(\nu,y)}\,.$

$$\gamma_{(\mu,x)(\nu,y)} = k_{\varphi\alpha}\delta^{kl}\left[\tilde{\mathcal{D}}^{\varphi}_{\;\mu\,k}(A^*(\mathbf{x}))\,\mathcal{D}^{\alpha}_{\;\nu\,l}(A^*(\mathbf{y}))\delta^3(\mathbf{x}-\mathbf{y})\right].$$

...principal bundle coordinates in Yang-Mills

The mechanical connection $\mathcal{A}^{\sigma}_{P}$ is defined as follows

$$\mathcal{A}^{\sigma}_{P}(Q^{*}) = \gamma^{\sigma\,\mu}(Q^{*})\, K^{R}_{\mu}(Q^{*})\, G_{RP}(Q^{*})\,.$$

In the Yang-Mills fields it is the Coulomb connection $\mathcal{A}^{\alpha}_{B}$:

$$\mathcal{A}^{(\alpha,x)}{}_{(\beta,j,y)} = \left[\mathcal{D}^{\varphi}{}_{\mu j}(A^{*}(\mathbf{y}))\gamma^{(\alpha,x)\,(\mu,y)}\right]\, k_{\varphi\beta}\,.$$

Horizontal Lagrange-Poincaré equations

$G_{AB} = \delta_{AB}$ and the Killing identities $K^{F}_{\sigma}\mathcal{F}^{\nu}_{EF} = 0 \rightarrow$

$$\frac{d\,\dot{Q}^{*A}}{dt} + {}^{\mathrm{H}}\Gamma^{A}_{BC}\dot{Q}^{*B}\dot{Q}^{*C} + G^{AS}N^{F}_{S}\mathcal{F}^{\nu}_{EF}\dot{Q}^{*E}p_{\nu} + \frac{1}{2}G^{AE}\,(\mathcal{D}_{E}\gamma^{\kappa\sigma})p_{\sigma}p_{\kappa}$$
$$+G^{AD}\partial_{D}V = 0,$$

Using the correspondence

$$A \to (\alpha, i, x); \;\; \mu \to (\mu, u); \; \dots \text{ etc} \, .$$

$$\dot{Q}^{*B}(t) \to \frac{d}{dt} A^{*(\sigma,j,y)}(t) \equiv \frac{d}{dt} A^{*\sigma j}(\mathbf{y}, t) \equiv \dot{A}^{*\sigma j}(\mathbf{y}, t),$$

and similar substitutions in all variables of finite-dimensional equations, together with the functional counterparts of the finite-dimensional expressions, it is possible to get the horizontal and vertical equations of motion for our Yang-Mills dynamical system.

The horizontal equation of motion:

$$
\frac{d}{dt}\dot{A}^{*\alpha i}(\mathbf{x},t) + \Big(-2\, c^{\alpha}_{\epsilon\beta}\, \dot{A}^{*\epsilon i}(\mathbf{x},t) \int d\mathbf{y}\, \mathcal{A}^{(\beta,x)}_{(\sigma,j,y)} \dot{A}^{*\sigma j}(\mathbf{y},t)
$$

$$
+ c^{\alpha}_{\mu\beta} \int d\mathbf{y} d\mathbf{z}\, \mathcal{A}^{(\beta,x)}_{(\epsilon,k,z)} \Big[\mathcal{D}^{\mu i}_{\nu}(A^*(\mathbf{x},t)) \mathcal{A}^{(\nu,x)}_{(\sigma,j,y)} \Big]\, \dot{A}^{*\sigma j}(\mathbf{y},t)\, \dot{A}^{*\epsilon k}(\mathbf{z},t) \Big)
$$

$$
+'' \mathcal{F} - \text{terms}''
$$

$$
\int d\mathbf{u}\, d\mathbf{z} \Big(-c^{\alpha}_{\mu\nu}\, \gamma^{(\sigma,z)\,(\mu,x)} \Big[\mathcal{D}^{\nu i}_{\epsilon}(A^*(\mathbf{x},t)) \gamma^{(\kappa,u)\,(\epsilon,x)} \Big]
$$

$$
+ c^{\sigma}_{\varphi\mu}\, \gamma^{(\mu,z)\,(\kappa,u)} \Big[\mathcal{D}^{\alpha i}_{\epsilon}(A^*(\mathbf{x},t)) \gamma^{(\varphi,z)\,(\epsilon,x)} \Big] \Big)\, p_{\kappa}(\mathbf{u},t)\, p_{\sigma}(\mathbf{z},t)
$$

$$
+ G^{(\alpha,i,x)(\gamma,m,y)} \frac{\delta}{\delta A^{(\gamma,m,y)}}\, V[A]\Big|_{A=A^*} = 0,
$$

$$
G^{(\alpha,i,x)(\gamma,m,y)} \frac{\delta}{\delta A^{(\gamma,m,y)}}\, V[A]\Big|_{A=A^*} = 2\mathcal{D}^{\alpha}_{\beta j}(A^*(\mathbf{x},t))(F^{\beta})^{ij}(\mathbf{x},t).
$$

The vertical equation of motion:

$$\frac{d}{dt}\,p_\sigma(\mathbf{x},t) - c^\kappa_{\varphi\sigma}\,p_\kappa(\mathbf{x},t)\int d\mathbf{y}\,\mathcal{A}^{(\varphi,x)}_{\;\;(\beta,j,y)}\,\dot{A}^{*\beta j}(\mathbf{y},t)$$
$$-c^\varphi_{\sigma\epsilon}\,p_\varphi(\mathbf{x},t)\int d\mathbf{y}\,\gamma^{(\epsilon,x)\,(\mu,y)}\,p_\mu(\mathbf{y},t) = 0.$$

The transition to equilibrium equations is achieved by using the substitution $\dot{A}^{*\alpha i} = 0$ in these equations.

The equilibrium equations

$$\int d\mathbf{u}\, d\mathbf{z} \Big(-c^{\alpha}_{\mu\nu}\, \gamma^{(\sigma,z)\,(\mu,x)} \left[\mathcal{D}^{\nu i}_{\epsilon}(A^*(\mathbf{x}))\gamma^{(\kappa,u)\,(\epsilon,x)}\right] + c^{\sigma}_{\varphi\mu}\, \gamma^{(\mu,z)\,(\kappa,u)} \left[\mathcal{D}^{\alpha i}_{\epsilon}(A^*(\mathbf{x}))\gamma^{(\varphi,z)\,(\epsilon,x)}\right]\Big)\, p_{\kappa}(\mathbf{u})\, p_{\sigma}(\mathbf{z}) = -2\mathcal{D}^{\alpha}_{\beta j}(A^*(\mathbf{x}))(F^{\beta})^{ij}(\mathbf{x})$$

$$-c^{\varphi}_{\sigma\epsilon}\, p_{\varphi}(\mathbf{x}) \int d\mathbf{y}\, \gamma^{(\epsilon,x)\,(\mu,y)}\, p_{\mu}(\mathbf{y}) = 0.$$

The solutions of the second equation are now the eigenfunctions of the Green function $\gamma^{(\epsilon,x)\,(\mu,y)}$ of the operator $(\tilde{\mathcal{D}}\,\mathcal{D})_{\mu\nu}$.

$$\gamma_{(\mu,x)(\nu,y)}\, \gamma^{(\nu,y)(\sigma,z)} = \delta^{(\sigma,z)}_{(\mu,x)} \equiv \delta^{\sigma}_{\mu}\, \delta^{3}(\mathbf{z}-\mathbf{x})\,.$$

That is,

$$k_{\varphi\alpha}\, \delta^{kl}\, \tilde{\mathcal{D}}^{\varphi}_{\mu\,k}(A^*(\mathbf{x}))\, \mathcal{D}^{\alpha}_{\nu\,l}(A^*(\mathbf{x}))\, \gamma^{(\nu,x)(\sigma,z)} = \delta^{\sigma}_{\mu}\, \delta^{3}(\mathbf{z}-\mathbf{x})\,.$$

(It is also necessary to impose the corresponding boundary conditions.)

Concluding remarks

1. The main fundamental difference from the finite-dimensional case is that in a pure Yang-Mills dynamical system there can in principle be a countable number of the relative equilibria. Apparently, this also occurs in other gauge systems.

2. An important question arises: what role do these additional equilibria play in the quantization of gauge systems?

3. For $p_\mu = 0$, the classical equation describes the motion of the reduced system given on the orbit space.
If $p_\mu \neq 0$, are we dealing with a description of some excitation of the system ?

Interaction of Yang-Mills fields with scalar fields – the model mechanical system

$$L = -\frac{1}{4g_0^2}F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{2}(\nabla_\mu\varphi^a)(\nabla^\mu\varphi^a)$$

The gauge $A_0^a = 0$ leads to

$$L = \frac{1}{2g_0^2}(\partial_0 A_i^a)^2 + \frac{1}{2}(\partial_0\varphi_i^a)^2 - -\frac{1}{4g_0^2}F^a_{ij}F^{a\,ij} - \frac{1}{2}(\nabla_i\varphi^a)(\nabla^i\varphi^a)$$

1. The rest gauge invariance with respect to $g(\bar{x})$.
2. The kinetic part of L is a sum of two quadratic form

So, the mechanical model may be a system which describes a motion of two interacting scalar particles on a special configuration space with a symmetry: the product of two spaces. The first space is a finite-dimensional Riemannian manifold, the second space is a finite-dimensional vector space.

The Riemannian metric of the manifold is

$$ds^2 = G_{AB}(Q)dQ^A dQ^B + G_{mn} df^m df^n,$$

The group acts isometrically on the manifold : $(p, v)g = (pg, g^{-1}v)$.

$$\tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^n = \bar{D}^n_m(g) f^m.$$

$\bar{D}^n_m(g) \equiv D^n_m(g^{-1})$, and reps $D^n_m(g)$ of $\mathcal{G}$ acts on the vector space V. Now $\pi : \mathcal{P} \times V \to \mathcal{P} \times_{\mathcal{G}} V$ means that locally $\pi : (p, v) \to [p, v]$, where $[p, v]$ is the equivalence class formed by the equivalence relation $(p, v) \sim (pg, g^{-1}v)$. The local section $\tilde{\sigma}_i$ of this bundle, $\pi \cdot \tilde{\sigma}_i = \mathrm{id}$, is the map which sends $[p, v]$ to some element $(\tilde{p}, \tilde{v}) \in \mathcal{P} \times V$. The section $\tilde{\sigma}_i$ is given by

$$\tilde{\sigma}_i([p, v]) = (\sigma_i(x), a(p)v),$$

where σ_i is a local section of $\mathrm{P}(\mathcal{M}, \mathcal{G})$, $\sigma_i : U_i \to \pi_{\mathrm{P}}^{-1}(U_i)$, $x = \pi_{\mathrm{P}}(p)$ and $a(p)$ is the group element defined by $p = \sigma_i(x)a(p)$.

The Lagrange-Poincaré equations

$$N^A_B\frac{d\omega^B}{dt}+N^A_R\,{}^{\mathrm{H}}\tilde{\Gamma}^R_{\tilde{B}\tilde{M}}\omega^{\tilde{B}}\omega^{\tilde{M}}+$$
$$G^{EF}N^A_E N^{\tilde{R}}_F\Big[\mathcal{F}^{\alpha}_{\tilde{Q}\tilde{R}}\omega^{\tilde{Q}}p_{\alpha}+\frac{1}{2}(\mathcal{D}_{\tilde{R}}d^{\kappa\sigma})p_{\kappa}p_{\sigma}+V_{,\tilde{R}}\Big]=0,$$

$$N^r_B\frac{d\omega^B}{dt}+\frac{d\omega^r}{dt}+N^r_{\tilde{R}}\,{}^{\mathrm{H}}\tilde{\Gamma}^{\tilde{R}}_{\tilde{A}\tilde{B}}\omega^{\tilde{A}}\omega^{\tilde{B}}+G^{EF}N^r_F N^{\tilde{R}}_E\Big[\mathcal{F}^{\alpha}_{\tilde{Q}\tilde{R}}\omega^{\tilde{Q}}p_{\alpha}+$$
$$\frac{1}{2}(\mathcal{D}_{\tilde{R}}d^{\kappa\sigma})p_{\kappa}p_{\sigma}+V_{,\tilde{R}}\Big]+G^{rm}\Big[\mathcal{F}^{\alpha}_{\tilde{Q}m}\omega^{\tilde{Q}}p_{\alpha}+\frac{1}{2}(\mathcal{D}_m d^{\kappa\sigma})p_{\kappa}p_{\sigma}+V_{,m}\Big]=0.$$

$$\frac{dp_{\beta}}{dt}+c^{\nu}_{\mu\beta}d^{\mu\sigma}p_{\sigma}p_{\nu}-c^{\nu}_{\sigma\beta}\mathcal{A}^{\sigma}_{\tilde{E}}\omega^{\tilde{E}}p_{\nu}=0.$$

$$\begin{aligned} d_{\mu\nu}(Q^*,\tilde{f}) &= K^A_{\mu}(Q^*)G_{AB}(Q^*)K^B_{\nu}(Q^*)+K^m_{\mu}(\tilde{f})G_{mn}K^n_{\nu}(\tilde{f}) \\ &\equiv \gamma_{\mu\nu}(Q^*)+\gamma'_{\mu\nu}(Q^*). \end{aligned}$$

The equations for the relative equilibrium

$$\begin{cases} N_B^R\Big(\dfrac{1}{2}(\mathcal{D}_R d^{\kappa\sigma})p_\kappa p_\sigma + V_{,R}\Big) = 0 \\ \frac{1}{2}(\mathcal{D}_m d^{\kappa\sigma})p_\kappa p_\sigma + V_{,m} = 0. \end{cases}$$

$$c^{\nu}_{\mu\beta} d^{\mu\sigma} p_\sigma p_\nu = 0.$$