Quartet-modified gravity, scalar-graviton dark substance and vacuum energy screening: Weyl transverse gravity vs. General Relativity

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Abstract

In the frameworks of the effective field theory of the metric gravity modified by a scalar quartet as the distinct dynamical coordinates, the extension of gravity through a massive scalar graviton in addition to the massless tensor one is consistently exposed. The field equations for the two generic realizations of such an extension originating from the classically equivalent prototype theories – General Relativity (GR) and its Weyl transverse (WTDiff) alternative – are derived and shown to be, generally, non-equivalent, with the pure gravity case manifesting this explicitly in detail. A generic splitting of the cosmological constant onto the gravitating and non-gravitating parts, resulting in a (partial) screening of the respective vacuum energy, is considered. It is stressed that possessing by a number of the attractive properties the WTDiff gravity should serve at least on par with GR as a prototype theory when looking for the next-to-GR extended theory of gravity, with an emergent scalar-graviton dark substance and a screening of the vacuum energy.

Keywords: modified gravity, WTDiff gravity, cosmological constant, vacuum energy.

1 Introduction

The contemporary Cosmology Standard Model or, more particularly, the ΛCDM model accumulates the present-day state-of-the-art for description of the evolution of the Universe.[1] According to its very name, the model incorporates such new ingredients of cosmology as the cosmological constant (CC) Λ comprising at present about 70% of the partial energy density of the Universe, and a (cold) dark matter (DM) comprising about 25% of such an "energy budget". At that, DM serves as a corner-stone to build the dark halos for the galaxies and cluster of galaxies. Being extremely economic in its basic concepts, the given model still shows an impressive success in describing the wide variety of the observational data. Nevertheless, some arguments (though predominantly of the theoretical origin) may imply a necessity of going eventually beyond ΛCDM.[2]

One of such an arguments is provided by the so-called CC/vacuum energy problem. First of all, why is Λ, though being rather large in its absolute value, still unnaturally small compared to what might be expected in the framework of the General Relativity (GR) as the effective field theory (EFT)? Secondly, why the classical Λ remains stable with account for the quantum corrections? And at last, why CC starts to manifest itself only rather lately, or as it sometimes is stated, "why now"? Though causing no principle difficulty phenomenologically, the CC problem may present theoretically the greatest challenge for the fundamental physics.[3]

[1]For a concise exposition of all the relevant topics, see [1].

[2]For ΛCDM and beyond, see, e.g., [2, 3].

[3]For the CC (or, more generally, dark energy (DE)) problem, see, e.g, [4]–[6].

Another tantalizing problem for gravity and cosmology is the DM one: what is the real nature of DM, especially in relation with the Particle Standard Model? Being not as crucial as CC for the theoretical consistency of the Cosmology Standard Model, DM causes still definite tension within GR.[4] Thus, though GR is up to now in a rather solid shape, nevertheless some its modification/extension may be in order.[5] Moreover, the DE and/or DM problems may even be the heralds of the future crucial changes in the present-day paradigms for gravity and cosmology.[6]

In this vein, in [10]–[12] there was proposed EFT of the so-called *quartet-modified/quartet-metric* gravity, the latter being based on the three following physical concepts.[7] First, in addition to the dynamical metric there exist in spacetime some distinct dynamical coordinates (associated ultimately with the vacuum) defined by a scalar quartet.[8] Second, the scalar quartet plays the role of the Higgs-like field for gravity. Third, the emergent additional gravity fields serve as the dark components (DM, DE, etc.) of the Universe. The proposed theory is originally invariant under the general diffeomorphisms (GDiff's) on the extended set of the fourteen fields. After fixing a background some gauge components contained in metric become physical through absorbing (a part of) the scalar quartet. Generically, such a theory describes in a completely dynamical, GDiff invariant and generally-covariant (GC) fashion the (massive) scalar and/or tensor (or vector) "gravitons" (though the vector one may be, in fact, unphysical). By this token, the mere admixture to metric of the scalar quartet may a priori result in a wide variety of the particular realizations of the generic quartet-metric EFT, with an extremely rich spectrum of the emergent physical phenomena beyond GR. To tame the ensuing ambiguities, as the most simple and natural version of the extended next-to-GR gravity there may be adopted the scalar-reduced quartet-metric gravity, with only a massive scalar graviton in addition to the conventional (massless) tensor one of GR.[9]

But even if one adheres to the concept of the scalar graviton, there still remains the wide residual freedom in choosing a preferred mode of its realization. Besides the evident ambiguity in choosing the particular Lagrangian there is also an ambiguity in choosing the prototype/"graft" theory of gravity undergoing the extension. Namely, one can choose either (i) GR with the GDiff invariance, or (ii) Weyl transverse/WTDiff gravity which, though restricted by the transverse Diff's (TDiff's), still allows for more gauge transformations – the local scale/Weyl ones.[10] The two prototype theories – GR and WTDiff gravity – prove to be classically equivalent (under the covariant conservation of the energy-momentum tensor), both containing the ten dynamical metric fields undergoing the four-parameter gauge transformations and possessing only by the two physical components describing the massless tensor graviton. What concerns CC, in GR the quantum CC problem is inborn as a manifestation of the longitudinal gravity component. On the contrary, in the WTDiff gravity the Lagrangian CC is, by the very construction, irrelevant, being substituted by an integration constant. The latter proves to be stable due to the Weyl invariance against the quantum corrections [25, 26]. Nevertheless, though equivalent classically as the prototype theories, GR and the WTDiff gravity may result in the non-equivalent extended theories already on the classical (moreover, on quantum) level. With this in mind, in the present paper we study the two above-mentioned alternatives on par in the context of the scalar-graviton dark substance with the ensuing vacuum energy screening.

[4] For DM, cf, e.g., [1].

[5] For the modified and extended theories of gravity beyond GR, cf., e.g., [7], [8].

[6] For importance of the new paradigms when accumulating the problems within the old ones, cf., e.g., [9].

[7] Such a gravity modification is allowed in the framework of EFT to be taken on its own ad hoc. Nevertheless, it admits a partial justification and refinement in the framework of a more fundamental nonlinear model [13].

[8] This concerns the four spacetime dimensions, with the proliferation to an arbitrary case being straightforward. Not to be fixed to the four spacetime dimensions, the respective theory had better be referred to as the *coordinate-modified* gravity.

[9] An earlier extension of GR due to the author [14], containing the scalar graviton with a non-dynamical scalar density for reconstructing GC, is naturally developed in the completely dynamical frameworks here.

[10] For the WTDiff gravity as a viable alternative to GR, see, e.g., [17]–[24].

In Section 2, the basics of the quartet-metric gravity as EFT is recapitulated, with a conceivable splitting of CC onto the two parts – gravitating and non-gravitating – being emphasized. In Section 3, the consistent scalar-graviton reduction of the generic quartet-metric gravity is presented starting from the first principles of the latter. A more restrictive model interpolating between GR and WTDiff gravity as the classically equivalent prototype theories resulting in the two non-equivalent extended theories is presented. The gravitational gauge symmetry differing these marginal cases is studied for both a dynamical and non-dynamical scalar density entering the composite scalar-graviton field. In Section 4, such a scalar-graviton reduction is studied more particularly for GR as a prototype theory, while in Section 5 the same is done for its WTDiff alternative, with the non-equivalence of the two approaches being shown explicitly in the pure gravity case. In Summary, the two alternatives are compared in respect to the emergent scalar-graviton dark substance and the (partial) screening of the proper vacuum energy. The conceivable advantages of the WTDiff extension are advocated, and a prior necessity to account for both alternatives when going beyond GR, to ultimately choose the most relevant one (if any) as the next-to-GR extended theory of gravity, is stressed.

2 Quartet-modified gravity

2.1 Generic case

The EFT of the quartet-metric/quartet-modified gravity [10]–[13] is generically defined by a GC scalar action

$$S = \int \mathcal{L}_G(g_{\mu\nu}, \Omega^a)\, d^4x, \tag{1}$$

with a Lagrangian scalar density $\mathcal{L}_G$ dependent on the two basic fields as the functions of the arbitrary observer's coordinates x^μ ($\mu = 0, \dots, 3$): a symmetric tensor field $g_{\mu\nu}(x)$, (with $g \equiv \det(g_{\mu\nu}) < 0$) and a quartet of the scalar fields $\Omega^a(x)$ ($a = 0, \dots, 3$. More particularly, let $a, b, \dots$ be the indices of the global Lorentz symmetry of the reparametrizations $\Omega^a \to \Lambda^a{}_b \Omega^b$, $\Lambda \in SO(1,3)$, possessing the invariant Minkowski symbol η_{ab}. By default, the signatures of $g_{\mu\nu}$ and η_{ab} are chosen to coincide. Assume moreover that the scalar fields Ω^a admit the global (not related to the spacetime) Poincare reparametrizations composed of the Lorentz ones and shifts $\Omega^a \to \Omega^a + C^a$, with the arbitrary constant parameters C^a. Due to the postulated GC and global Poincare invariance, Ω^a should, in fact, enter the action through an auxiliary *quasi-affine* metric

$$\omega_{\mu\nu} \equiv \partial_\mu \Omega^a \partial_\nu \Omega^b \eta_{ab}, \tag{2}$$

Consider now a connected spacetime region (an "affine patch") where $\det(\partial\Omega^a/\partial x^\mu) \neq 0$ or ∞ allowing thus to invert the dependence $\Omega^a = \Omega^a(x)$ to $x^\mu = x^\mu(\Omega)$. By this token, we can cover the spacetime manifold M_4 in a patch-wise fashion by some distinct dynamical coordinates – the *quasi-affine* ones $\mathring{x}^\alpha = \delta^\alpha_a \Omega^a(x)$, $\alpha = 0, \dots, 3$, (with an inverse $x^\mu = x^\mu(\mathring{x})$).[11] For a (patch-wise) invertibility of the spacetime coordinate transformations, $\mathring{x}^\alpha = \delta^\alpha_a \Omega^a(x)$, the Jacobian should fulfill the condition $\det(\partial_\mu \Omega^a) \neq 0$ and hence

$$\omega \equiv \det(\omega_{\mu\nu}) = \det(\partial_\mu \Omega^a)^2 \det(\eta_{ab}) < 0, \tag{3}$$

implying the non-degeneracy of the quasi-affine metric $\omega_{\mu\nu}$, having thus an inverse $\omega^{-1\mu\nu}$. Operationally, the quasi-affine coordinates $\mathring{x}^\alpha$ are distinct by the fact that under using them the

[11]Sewing the borders of the affine patches, as well as inclusion of the singular spacetime points (if any), where the invertibility of $\mathring{x}^\alpha(x)$ breaks down, should be treated separately.

quasi-affine metric gets Minkowskian form, $\omega_{\alpha\beta}(\mathring{x}) \equiv \eta_{\alpha\beta}$ (respectively, $\omega^{-1\alpha\beta}(\mathring{x}) \equiv \eta^{\alpha\beta}$). Physically, the coordinates $\mathring{x}^\alpha$ may be postulated as those comoving with the vacuum, the latter treated ultimately as a dynamical system on par with all the dynamical fields including metric $g_{\mu\nu}$.[12]

Altogether, the action of the quartet-modified metric gravity may most generally be rewritten in an entirely spacetime form as

$$S = \int \mathcal{L}_G(g_{\mu\nu}, \omega_{\mu\nu}, g, \omega) d^4x. \tag{4}$$

At that, as the basic dynamical variables remain still the scalar quartet Ω^a in the line with the "bare" metric $g_{\mu\nu}$. The Lagrangian density $\mathcal{L}_G$ may further be decomposed as

$$\mathcal{L}_G = L_G(g_{\mu\nu}, \omega_{\mu\nu}, g/\omega)\, \mathcal{M}(g, \omega), \tag{5}$$

with a GC scalar Lagrangian L_G supplemented by a spacetime measure $\mathcal{M}$. The latter is a GC scalar density of the proper weight entering the spacetime volume element $dV = \mathcal{M} d^4x$ to make the latter a GC scalar. In view of $\omega \neq 0$, the sign of $\sqrt{-\omega}$ is (patch-wise) preserved and we can put $\sqrt{-\omega} > 0$. A priori, the measure is defined up to a scalar function $\varphi_{\mathcal{M}}(g/\omega)$, which may be attributed, if desired, to L_G. Thus, with the proper redefinition of L_G, the measure may equivalently be chosen either as $\sqrt{-g}$ or $\sqrt{-\omega}$, depending on the context. Altogether, prior to fixing the Lagrangian we can put without loss of generality, say,

$$S = \int L_G(g_{\mu\nu}, \omega_{\mu\nu}, \omega/g) \sqrt{-g}\, d^4x. \tag{6}$$

Generically, the Lagrangian L_G describes the multi-component gravity mediated by the massive (one-component) scalar, (five-component) tensor or (three-component) vector gravitons contained in the metric (a part of them may become unphysical), with Ω^a serving ultimately as a gravity counterpart of the Higgs field which provides the four missing additional components.[13] The Lagrangian L_G quadratic in the first derivatives of metric is constructed in [10]. A more general Lagrangian is discussed in [11, 12]. Imposing on the parameters of L_G the "natural" (in a technical sense) restrictions we can exclude the vector graviton, as the most "suspicious, leaving just the massive scalar graviton in addition to the massless tensor one, the case to be treated in what follows.

2.2 Cosmological constant splitting

In the framework of the quartet-modified gravity it is always possible to include in $\mathcal{L}_G$ eq. (5) the field-independent pure-measure contribution

$$\Delta\mathcal{L}_\Lambda = -\kappa_g^2(\Lambda_g \sqrt{-g} + \Lambda_\omega \sqrt{-\omega}) \tag{7}$$

containing, instead of a single conventional CC Λ, the two, generally, independent CC's: a "gravitating/Riemannian" Λ_g and a "non-gravitating/quasi-affine" Λ_ω of the dimensions mass squared. Such a conceivable CC splitting to account for the vacuum energy is a generic trait of the quartet-modified gravity compared to GR and its direct siblings.

[12] More particularly, we should put at the quantum level $\Omega^a = \bar{\Omega}^a + \chi^a$, with the background $\bar{\Omega}^a \equiv \delta^a_\alpha \mathring{x}^\alpha$ and $\chi^\alpha = \delta^\alpha_a \chi^a$ being the quantum fluctuations around the background, similarly to the respective decomposition of the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. In essence, this may be considered as a kind of the spacetime quantization.

[13] Namely, instead of $\omega_{\mu\nu}$ we could equivalently use a Higgs-like field $H^\mu{}_\nu = g^{\mu\lambda}\omega_{\lambda\nu}$ (so that $\det(H^\mu{}_\nu) = \omega/g$), with $H^\mu{}_\nu$ defining, in fact, the derivativeless/potential part $V_G(H)$ of L_G [13].

3 Scalar-graviton reduction

3.1 Generic case

The ensuing consideration significantly simplifies (remaining still rather rich of a new content) in a reduced case of the quartet-metric gravity given by the Lagrangian L_G dependent on $\omega_{\mu\nu}$ exclusively through its determinant ω as

$$S = \int \mathcal{L}_{gs}(g_{\mu\nu}, g, \omega) d^4x = \int L_{gs}(g_{\mu\nu}, \omega/g)\sqrt{-g} d^4x. \tag{8}$$

The ratio ω/g may equivalently be substituted by

$$\sigma \equiv \ln \sqrt{-g}/\sqrt{-\omega}, \tag{9}$$

so that one can always put

$$\mathcal{L}_{gs}(g_{\mu\nu}, g, \omega) = L_{gs}(g_{\mu\nu}, \sigma)\sqrt{-g}. \tag{10}$$

With ω having the same weight as g under the general coordinate transformations, σ is a true GC scalar with the normalization $(\sigma|_{g=\omega} = 0)$. Stress that σ, to be called the scalar graviton, has a composite nature, ultimately distinguishing such a scalar field from the elementary one. Eqs. (3) and (9), with $g_{\mu\nu}$ and Ω^a as the independent gravity field variables, are the key ingredients of the completely dynamical theory of the scalar graviton.

Constructing the most general theory of the scalar graviton was undertaken in [12]. In what follows, we concentrate ourselves on a more restrictive but, hopefully, still promising reduction of the general theory. More particularly, let us introduce the conformally rescaled metric

$$\tilde{g}_{\mu\nu} \equiv \tilde{\varphi}_g(\sigma) g_{\mu\nu}, \tag{11}$$

with $(-\det(\tilde{g}_{\mu\nu}))^{1/2} \equiv \sqrt{-\tilde{g}} = \tilde{\varphi}_g^2 \sqrt{-g}$, and $\tilde{g}^{-1\mu\nu} = \tilde{\varphi}_g^{-1}(\sigma) g^{\mu\nu}$ being an inverse of $\tilde{g}_{\mu\nu}$. By this token, the most general Lagrangian density for the pure scalar-tensor quartet-modified gravity may be presented equivalently as

$$\tilde{\mathcal{L}}_{gs} = \tilde{L}_{gs}(\tilde{g}_{\mu\nu}, \sigma)\sqrt{-\tilde{g}} \tag{12}$$

to be generally understood in what follows.[14,15]

Now, in the presence of matter the respective Lagrangian density looks like

$$\tilde{\mathcal{L}}_{gsm} = \tilde{L}_{gsm}(\tilde{g}_{\mu\nu}, \sigma, \phi_I)\sqrt{-\tilde{g}}. \tag{13}$$

where ϕ_I is a generic matter field. The looked-for theory for the scalar graviton [12] may be expressed as a modification of a general scalar-tensor (matterless) theory for the metric $g_{\mu\nu}$ and a scalar field σ [15], [16] (in the four spacetime dimensions) after imposing the constraint (9). At that, the effective metric $\tilde{g}_{\mu\nu}$ remains unspecified due to the residual conformal redefinitions. To abandon such an ambiguity, below is considered a restricted but still rather general class of the scalar-tensor theories embodying, hopefully, a looked-for one.

[14] By default, the spacetime tensor indices are assumed to be manipulated by the effective metrics $\tilde{g}_{\mu\nu}$ and $\tilde{g}^{-1\mu\nu}$, and not by the bare ones $g_{\mu\nu}$ and $g^{-1\mu\nu} \equiv g^{\mu\nu}$ (if not stated otherwise).

[15] Allowing the dependence on the whole $\omega_{\mu\nu}$, the effective metric in the quartet-modified gravity could be taken as the *"disformal"* one $\tilde{g}_{\mu\nu} \equiv \tilde{\varphi}_g(\sigma) g_{\mu\nu} + \tilde{\varphi}_\omega(\sigma)\omega_{\mu\nu}$, with the two scalar functions $\tilde{\varphi}_g(\sigma)$ and $\tilde{\varphi}_\omega(\sigma)$ bound to assure the non-degeneracy of $\tilde{g}_{\mu\nu}$.

3.2 Factorized model

Postulate the Lagrangian sufficiently generally in the partially factorized form corresponding to the effective pure-tensor gravity and the rest:

$$\tilde{L}_{gsm} = \tilde{L}_g(\partial_\kappa \tilde{g}_{\mu\nu}, \tilde{g}_{\mu\nu}) + \tilde{L}_{sm}(\partial_\kappa \sigma, \partial_\kappa \phi_I, \tilde{g}_{\mu\nu}, \sigma, \phi_I), \tag{14}$$

subsequently fixing the form of $\tilde{L}_g$. Minimally, we can put

$$\tilde{L}_g = -\frac{1}{2}\kappa_g^2 R(\tilde{g}_{\mu\nu}), \tag{15}$$

with $\kappa_g = 1/\sqrt{8\pi G_N}$ being the truncated Planck scale, $\tilde{R} \equiv R(\tilde{g}_{\mu\nu}) = \tilde{g}^{-1\kappa\lambda} R_{\kappa\lambda}(\tilde{g}_{\mu\nu})$ the Ricci scalar and $R_{\mu\nu}(\tilde{g}_{\mu\nu})$ the Ricci tensor. After picking-up the pure-tensor gravity Lagrangian $\tilde{L}_g$ the only freedom remains in the scalar-matter Lagrangian $\tilde{L}_{sm}$, being a priori an arbitrary function of its arguments.

More particularly, picking-up the dependence on the derivatives of the fields, we could further assume the factorization:

$$\tilde{L}_{sm} = \tilde{L}_s(\partial_\kappa \sigma, \tilde{g}_{\mu\nu}, \sigma, \phi_I) + \tilde{L}_m(\partial_\kappa \phi_I, \tilde{g}_{\mu\nu}, \phi_I, \sigma) - V_{sm}(\sigma, \phi_I). \tag{16}$$

In the second-derivative approximation the scalar-graviton Lagrangian may be taken in the most general quadratic form as

$$\tilde{L}_s = \frac{1}{2}\kappa_s^2 \tilde{\varphi}_s(\sigma, \phi_I)\tilde{g}^{-1\mu\nu}\partial_\mu \sigma \partial_\nu \sigma, \tag{17}$$

where $\kappa_s \ll \kappa_g$ is a scalar-graviton scale. If the kinetic profile function $\tilde{\varphi}_s$ is independent of ϕ_I eq. (17) implies a redefinition of the scalar-graviton field through

$$\sigma \to \tilde{\sigma} = \int^\sigma \tilde{\varphi}_s^{1/2}(\sigma')d\sigma'. \tag{18}$$

The kinetic matter Lagrangian $\tilde{L}_m$ remains still an arbitrary function of its arguments. The potential V_{sm} describes generically the masses of the fields and their interactions (incorporating, possibly, the spontaneous symmetry breaking).

3.3 GR-to-WTDiff interpolation model

3.3.1 Dynamical scalar density

To grasp the essence of the scalar-graviton extension, consider even more restrictive but still sufficiently general model given by the effective metric corresponding to $\tilde{\varphi}_g(\sigma) = e^{-\gamma\sigma/2}$ dependent on an arbitrary constant parameter γ, so that

$$\tilde{g}_{\mu\nu} \equiv e^{-\gamma\sigma/2} g_{\mu\nu} = (\omega/g)^{\gamma/4} g_{\mu\nu}, \tag{19}$$

with an inverse $\tilde{g}^{-1\mu\nu} = e^{\gamma\sigma/2} g^{\mu\nu} = (g/\omega)^{\gamma/4} g^{\mu\nu}$ and $\tilde{g} = g^{(1-\gamma)}\omega^\gamma$. A trait of GR is its gauge symmetry – GDiff – on which GR (and it direct siblings) are built. So, a concise way of treating a modified gravity beyond GR is to consider the modification of the respective gauge symmetry compared to the conventional GDiff as the reference one. At $\gamma \neq 0$, the effective metric $\tilde{g}_{\mu\nu}$ contains a priori the exhaustive number, eleven, of the independent variables. To reduce the latter ones to the conventional six for the tensor gravity, consider for the primary fields $g_{\mu\nu}$ and Ω^a the gauge transformations in terms of the five-parameter combined gauge symmetry consisting of GDiff's (the Lie derivatives) defined by an infinitesimal vector field $\xi^\mu(x)$ supplemented by the transformations defined by an infinitesimal scalar field $\zeta(x)$.

In these terms, an infinitesimal gauge transformation D (a combined Lie derivative) acting on the primary metric $g_{\mu\nu}$ is as follows:

$$
\begin{aligned}
Dg_{\mu\nu} &= g_{\mu\lambda}\partial_\nu\xi^\lambda + g_{\nu\lambda}\partial_\mu\xi^\lambda + \xi^\lambda\partial_\lambda g_{\mu\nu} + \zeta g_{\mu\nu} \\
&= g_{\mu\lambda}\nabla_\nu\xi^\lambda + g_{\nu\lambda}\nabla_\mu\xi^\lambda + \zeta g_{\mu\nu}, \\
D\sqrt{-g}/\sqrt{-g} &= 1/2\, g^{\mu\nu}Dg_{\mu\nu} = \partial_\lambda(\sqrt{-g}\xi^\lambda)/\sqrt{-g} + 2\zeta = \nabla_\lambda\xi^\lambda + 2\zeta,
\end{aligned} \tag{20}
$$

with ξ^λ and ζ the arbitrary vector and scalar, respectively, and ∇_λ being a covariant derivative with respect to $g_{\mu\nu}$. Proliferate D further on $\omega_{\mu\nu}$ as a conventional Lie derivative[16]

$$
\begin{aligned}
D\Omega^a &= \xi^\lambda\partial_\lambda\Omega^a \equiv \xi^\lambda\Omega^a_\lambda, \\
D\Omega^a_\mu = \partial_\mu(D\Omega^a) &= \Omega^a_\lambda\partial_\mu\xi^\lambda + \xi^\lambda\partial_\lambda\Omega^a_\mu, \\
D\omega_{\mu\nu} &= \eta_{ab}(\Omega^a_\mu D\Omega^b_\nu + \Omega^a_\nu D\Omega^b_\mu) \\
&= \omega_{\mu\lambda}\partial_\nu\xi^\lambda + \omega_{\nu\lambda}\partial_\mu\xi^\lambda + \xi^\lambda\partial_\lambda\omega_{\mu\nu},
\end{aligned} \tag{21}
$$

where use was made of $\partial_\mu\Omega^a_\lambda = \partial_\mu\partial_\lambda\Omega^a = \partial_\lambda\Omega^a_\mu$, so that

$$
D\sqrt{-\omega}/\sqrt{-\omega} = 1/2\,\omega^{-1\mu\nu}D\omega_{\mu\nu} = \partial_\lambda(\sqrt{-\omega}\xi^\lambda)/\sqrt{-\omega} = \xi^\lambda\partial_\lambda\ln\sqrt{-\omega} + \partial_\lambda\xi^\lambda. \tag{22}
$$

This results in the infinitesimal transformations of the effective fields σ and $\tilde{g}_{\mu\nu}$ at an arbitrary γ as follows:

$$
\begin{aligned}
D\sigma &= D\sqrt{-g}/\sqrt{-g} - D\sqrt{-\omega}/\sqrt{-\omega} = \xi^\lambda\partial_\lambda\sigma + 2\zeta, \\
D\tilde{g}_{\mu\nu} &= D(e^{-\gamma\sigma/2}g_{\mu\nu}) = \tilde{g}_{\mu\lambda}\partial_\nu\xi^\lambda + \tilde{g}_{\nu\lambda}\partial_\mu\xi^\lambda + \xi^\lambda\partial_\lambda\tilde{g}_{\mu\nu} + (1-\gamma)\zeta\tilde{g}_{\mu\nu} \\
&= \tilde{g}_{\mu\lambda}\tilde{\nabla}_\nu\xi^\lambda + \tilde{g}_{\nu\lambda}\tilde{\nabla}_\mu\xi^\lambda + (1-\gamma)\zeta\tilde{g}_{\mu\nu}
\end{aligned} \tag{23}
$$

(incorporating, in particular, the case with $\gamma = 0$ at $\tilde{g}_{\mu\nu} = g_{\mu\nu}$). In the above, $\tilde{\nabla}_\lambda$ means a covariant derivative with respect to $\tilde{g}_{\mu\nu}$.[17]

At $\gamma \neq 0$, the effective metric $\tilde{g}_{\mu\nu}$ depends on an extra variable which may be tried to be eliminated by the Weyl rescaling. But even if we do so for $g_{\mu\nu}$, the extra variable will reappear again in $\tilde{g}_{\mu\nu}$ through ζ. The only exception corresponds to $\gamma = 1$. In this case, the Lagrangian $\tilde{L}_g \sim R(\tilde{g}_{\mu\nu})$ describes on its own the pure tensor gravity invariant explicitly under the conventional GDiff. Such an extended gravity case, invariant under Weyl rescaling times GDiff, may thus be called the *WGDiff gravity*. Under WGDiff, according to (22) we can use first the longitudinal Diff to achieve, say, the canonical value $\omega = -1$ and then use WTDiff to eliminate the four components from $g_{\mu\nu}$, reducing the number of the independent components in $\tilde{g}_{\mu\nu}$ precisely to six. At that, according to (23) the inclusion of σ in an extended Lagrangian would explicitly violate WGDiff to GDiff as a maximal gauge symmetry compatible with the scalar graviton. One more marginal case corresponds to $\gamma = 0$, with the metric $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu}$ not containing an extra variable and henceforth not requiring the Weyl rescaling to eliminate it. In this case, the Lagrangian $L_g \sim R(g_{\mu\nu})$ is invariant precisely under GDiff representing explicitly the conventional tensor gravity. The addition of terms with σ, due to the implied $\zeta \equiv 0$, is still compatible with the same GDiff. Thus, we encounter the two conceivable patterns of the gauge symmetry for the theory of the scalar graviton with a dynamical ω: GDiff for both the pure-tensor and scalar-tensor gravity, or WGDiff for the pure-tensor gravity with the residual GDiff for the scalar-tensor gravity.

[16] The inclusion of the local gauge transformation $D\Omega^a \sim \zeta\Omega^a$ would violate the assumed global transformation symmetry for Ω^a, and so is not considered.

[17] In fact, the transformations with the infinitesimal ζ may be proliferated to the finite ones $g_{\mu\nu} \to Zg_{\mu\nu}$, $\sigma \to \sigma + 2\ln Z$ and $\tilde{g}_{\mu\nu} \to Z^{(1-\gamma)}\tilde{g}_{\mu\nu}$, with $Z \equiv e^\zeta$ being nothing but the Weyl rescaling factor.

3.3.2 Non-dynamical scalar density

For comparison, let now ω be a given non-dynamical scalar density a priori allowed in the interpolating model. To account for this, introduce the restricted infinitesimal gauge transformation $\mathring{D}$ putting by default $\mathring{D}\omega \equiv 0$ and $\mathring{D}g_{\mu\nu} \equiv Dg_{\mu\nu}$, so that

$$\begin{aligned}\mathring{D}\sigma &= D\sqrt{-g}/\sqrt{-g} = 1/2\, g^{\mu\nu} Dg_{\mu\nu} = \partial_\lambda(\sqrt{-g}\xi^\lambda)/\sqrt{-g} + 2\zeta \\ &= \xi^\lambda \partial_\lambda \sigma + \partial_\lambda(\sqrt{-\omega}\xi^\lambda)/\sqrt{-\omega} + 2\zeta,\end{aligned} \tag{24}$$

giving at an arbitrary γ:

$$\mathring{D}\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\lambda}\partial_\nu\xi^\lambda + \tilde{g}_{\nu\lambda}\partial_\mu\xi^\lambda + \xi^\lambda\partial_\lambda\tilde{g}_{\mu\nu} - \gamma/2\,\partial_\lambda(\sqrt{-\omega}\xi^\lambda)/\sqrt{-\omega}\,\tilde{g}_{\mu\nu} + (1-\gamma)\zeta\tilde{g}_{\mu\nu}. \tag{25}$$

It follows that in order that the Lagrangian $\tilde{L}_g \sim R(\tilde{g}_{\mu\nu})$ corresponds on its own to the pure-tensor gravity it is necessary to fulfill the constraint

$$(1-\gamma)\zeta - \gamma/2\,\partial_\lambda(\sqrt{-\omega}\xi^\lambda)/\sqrt{-\omega} = 0 \tag{26}$$

meaning vanishing of a γ-dependent combination of the longitudinal Diff and the Weyl rescaling. In particular, at $\gamma = 0$ this implies the contraction $\zeta \equiv 0$ under an arbitrary ξ^λ what in turn means GDiff, while at $\gamma = 1$ (putting for definiteness $\omega = -1$) there follows the contraction $\partial_\lambda\xi^\lambda \equiv 0$ under an arbitrary ζ meaning restriction to WTDiff. At that, under eq. (26) one gets

$$\begin{aligned}\mathring{D}\sigma &= \xi^\lambda\partial_\lambda\sigma + 1/(1-\gamma)\,\partial_\lambda(\sqrt{-\omega}\xi^\lambda)/\sqrt{-\omega}, \quad \text{at} \quad 0 \le \gamma < 1, \\ \mathring{D}\sigma &= \xi^\lambda\partial_\lambda\sigma + 2/\gamma\,\zeta, \quad \text{at} \quad 0 < \gamma \le 1,\end{aligned} \tag{27}$$

both expressions valid on par at the intermediate $0 < \gamma < 1$. It follows that the inclusion of terms with σ in an extended Lagrangian explicitly restricts the residual gauge symmetry to TDiff in any case – GDiff ($\gamma = 0$, $\omega = -1$) or WTDiff ($\gamma = 1$) – signifying the appearance of a scalar graviton in excess of the massless tensor one.[18,19]

Altogether, we can encounter the four viable gauge alternatives for describing the pure-tensor gravity in terms of $R(\tilde{g}_{\mu\nu})$ on its own: (i) the three-parameter TDiff for the nine independent metric variables under $g \equiv \omega$, with a non-dynamical ω; (ii) the four-parameter WTDiff (at $\gamma = 1$) for the ten independent metric variables, with a non-dynamical ω; (iii) the four-parameter GDiff (at $\gamma = 0$) for the ten (or eleven) independent gravity variables, with ω non-dynamical (or dynamical); and at last (iv) the five-parameter WGDiff (at $\gamma = 1$) for the eleven independent gravity variables, with a dynamical ω. These alternatives may be used as the prototype ones for the, generally, non-equivalent extensions of the pure-tensor gravity through the scalar graviton. In a wide perspective, the problem is which mode of the gravitational gauge symmetry – TDiff, WTDiff, GDiff or, most generally, WGDiff – to use for unifying gravity at the level of EFT with matter including an emergent scalar-graviton dark substance (if any).[20]

Below, the two marginal cases – GR and WTDiff – of the interpolating model extended through the scalar graviton with a dynamical ω, possessing the gauge symmetries, respectively, GDiff ($\gamma = 0$) and WGDiff ($\gamma = 1$) for the pure-tensor gravity and the residual GDiff for the scalar-tensor one are studied in detail.

[18] In fact, the terms with a derivative of σ are still invariant under a global shift symmetry. This may be used to justify the suppression of terms containing explicitly σ with no derivative.

[19] At a fixed non-dynamical ω (say, $\omega = -1$) the two marginal cases corresponding here to $\gamma = 0$ and $\gamma = 1$ are moreover argued to be the only ones describing the two-component massless tensor graviton in terms of the ten-component metric field, with the difference between the cases lying in the required gauge symmetry: GDiff vs. WTDiff [18].

[20] Generally, the scalar graviton is composite, $\sigma = \sigma(g/\omega)$. An exception corresponds to TDiff, with $g/\omega = 1$ implying the necessity of an elementary scalar field as a scalar graviton. In this respect, for a scale-invariant unification of Standard Model with gravity and DE in the frameworks of the unimodular/TDiff gravity extended by a dilaton, cf. [27].

4 Extending General Relativity

4.1 Basic formalism

As a reference case we choose $\gamma = 0$, with $\tilde{\varphi}_g \equiv 1$ and the effective metric $\tilde{g}_{\mu\nu} = g_{\mu\nu}$ coinciding with the bare one corresponding to GR as the prototype theory. With account for the basic variations

$$\begin{aligned} \delta\sqrt{-g}/\sqrt{-g} &= -1/2\, g_{\mu\nu}\delta g^{\mu\nu}, \\ \delta\sqrt{-\omega}/\sqrt{-\omega} &= 1/2\, \omega^{-1\kappa\lambda}\delta\omega_{\kappa\lambda}, \end{aligned} \tag{28}$$

where

$$\delta\omega_{\kappa\lambda} = \eta_{ab}(\Omega^a_\kappa \delta\Omega^b_\lambda + \Omega^a_\lambda \delta\Omega^b_\kappa), \tag{29}$$

we then get

$$\begin{aligned} \delta\sigma &= \delta\sqrt{-g}/\sqrt{-g} - \delta\sqrt{-\omega}/\sqrt{-\omega} \\ &= -1/2(g_{\mu\nu}\delta g^{\mu\nu} + \omega^{-1\kappa\lambda}\delta\omega_{\kappa\lambda}). \end{aligned} \tag{30}$$

In the above, $\omega^{-1\kappa\lambda} = \Omega^{-1}{}^{\kappa}_{a}\Omega^{-1}{}^{\lambda}_{b}\eta^{ab}$ is an inverse of $\omega_{\kappa\lambda}$, with $\Omega^{-1}{}^{\kappa}_{a} \equiv \partial x^\kappa/\partial\Omega^a$ being a tetrad inverse of $\Omega^a_\kappa \equiv \partial_\kappa \Omega^a$. By this token, adding $\Delta\mathcal{L}_\Lambda$ given by (7) to $\mathcal{L}_{gsm} = L_{gsm}\sqrt{-g}$, equating to zero the coefficients at the variations of the total Lagrangian density $\mathcal{L}_{\rm tot}$ with respect to the independent variations $\delta g^{\mu\nu}$, $\delta\omega_{\kappa\lambda}$ and $\delta\phi_I$ we get the system of the field equations (FEs) in a conventional notation as follows:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} - \Lambda_g g_{\mu\nu} = \frac{1}{\kappa_g^2}(T_{sm\mu\nu} + \Delta T_{sm\mu\nu}), \\ \nabla_\kappa\Big((\kappa_g^2\Lambda_\omega e^{-\sigma} + \frac{\delta L_{sm}}{\delta\sigma})\Omega^{-1}{}^{\kappa}_{a}\Big) &= 0, \\ \frac{\delta L_{sm}}{\delta\phi_I} &= 0, \end{aligned} \tag{31}$$

where δ/δ means a total variational derivative including a derivative with respect to the derivatives of the fields. In the above, one conventionally has

$$T_{sm\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L_{sm})}{\partial g^{\mu\nu}} = 2\frac{\partial L_{sm}}{\partial g^{\mu\nu}} - L_{sm}g_{\mu\nu} \tag{32}$$

as the total canonical energy-momentum tensor of the scalar graviton and matter, with ∂/∂ meaning a partial variational derivative, supplemented by

$$\Delta T_{sm\mu\nu} = -\frac{\delta L_{sm}}{\delta\sigma}g_{\mu\nu}. \tag{33}$$

The second FE of (31) restricts ω and σ, and the third FE clearly accounts for matter. Applying to the first FE of (31) the covariant derivative and using the truncated Bianchi identity, $\nabla_\mu(R^{\mu\nu} - 1/2\, Rg^{\mu\nu}) = 0$, we get the modified covariant conservation/continuity condition

$$\nabla_\nu(T^{\mu\nu}_{sm} + \Delta T^{\mu\nu}_{sm}) = 0 \tag{34}$$

for the total energy-momentum tensor (but, generally, not for $T_{sm\mu\nu}$ on its own). Stress that Λ_ω does not directly enter the tensor-gravity FE (31) (henceforth its name non-gravitating) in distinction from the gravitating Λ_g which does enter this FE.[21]

[21]For a non-gravitating vacuum energy associated with a dynamical non-geometrical measure in terms of a scalar quartet, cf., e.g., [28, 29]. For a spacetime volume element and a quartet of scalar fields, cf. also [30].

Note now, that the second FE for Ω^a may otherwise be considered as a projection of FE for the scalar graviton σ with a source determined by Λ_ω. Under a non-dynamical ω, in neglect by this FE $\delta L_{sm}/\delta\sigma$ (as well as Λ_ω) drops out. The remaining second FE of (31) determines in this case σ only up to an arbitrary non-dynamical scalar density ω, with the explicit violation of GDiff to the residual TDiff invariance. Under the dynamical ω, in the formal limit $\Lambda_\omega \to \infty$ the third FE due to $e^{-\sigma} = \sqrt{-\omega}/\sqrt{-g}$ factorizes as $\partial_\kappa(\sqrt{-\omega}\Omega^{-1}{}^\kappa_a) = 0$, with the (conceivably, large) CC Λ_ω dropping out from the classical FEs. Moreover, such a decoupling takes place exactly in a specific case of the scalar graviton satisfying the solution $\delta L_{sm}/\delta\sigma = Ce^{-\sigma}$, with a constant C (see, later).

Separating the tensor-gravity FE onto the transversal/traceless and longitudinal/trace parts we present these FE equivalently as

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{4}R\, g_{\mu\nu} - \frac{1}{\kappa_g^2}(T_{sm\mu\nu} - \frac{1}{4}T_{sm}\, g_{\mu\nu}) &= 0, \\
R + 4\Lambda_g + \frac{1}{\kappa_g^2}(T_{sm} + \Delta T_{sm}) &= 0,
\end{aligned}
\tag{35}
$$

where

$$
T_{sm} = T_{sm\mu\nu}g^{\mu\nu} = 2\frac{\partial L_{sm}}{\partial g^{\mu\nu}}g^{\mu\nu} - 4L_{sm}, \quad \Delta T_{sm} = -4\frac{\delta L_{sm}}{\delta\sigma}. \tag{36}
$$

When comparing the extensions to GR and the TDiff gravity, with the transversal parts of the tensor-gravity FEs looking similarly, the longitudinal ones are precisely those which differ the theories (see, later).

4.2 Lagrange multiplier formalism

4.2.1 Generic case

Basically, the independent field variables of gravity are assumed to be the "bare" metric $g_{\mu\nu}$ and the distinct dynamical coordinates (given by the quartet Ω^a), with the (composite) σ given by (9). Equivalently (at least, on the classical level), we can use an alternative formalism with an indefinite Lagrange multiplier by adding a constraint Lagrangian density $\Delta\mathcal{L}_\lambda$. With the latter adequately chosen, such a formalism allows to make the technical procedure simpler and the physics interpretation of the (formalism independent) results more transparent. To this end, choose the proper Lagrangian density as

$$
\Delta\mathcal{L}_\lambda = \lambda(\sqrt{-\omega} - e^{-\sigma}\sqrt{-g}) \tag{37}
$$

and treat the Lagrange multiplier λ and the scalar-graviton field σ as the two additional independent field variables. Varying now the total action independently with respect to λ, $g^{\mu\nu}$, σ and Ω^a we first get the relation $\sqrt{-\omega} = e^{-\sigma}\sqrt{-g}$, to be understood where necessary, followed by FEs for the tensor and scalar gravity, as well as for the quartet, respectively, as follows:

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda_g g_{\mu\nu} &= \frac{1}{\kappa_g^2}(T_{sm\mu\nu} + \lambda e^{-\sigma}g_{\mu\nu}), \\
\frac{\delta L_{sm}}{\delta\sigma} + \lambda e^{-\sigma} &= 0, \\
\nabla_\kappa\Big((\lambda - \kappa_g^2\Lambda_\omega)e^{-\sigma}\Omega^{-1}{}^\kappa_a\Big) &= 0,
\end{aligned}
\tag{38}
$$

with FE for matter remaining as before. Excluding λ we uniquely recover the original (λ-independent) form of FEs (31) (but not uniquely v.v.). Such a simple form of the derived FEs (38), being more useful for the practical purposes, clearly obligates due to the convenient choice of $\Delta\mathcal{L}_\lambda$.

4.2.2 Matterless case

Apply the Lagrange multiplier formalism to the pure-gravity matterless case, $L_{sm} = L_s$, when the whole consideration may be proceeded explicitly up to the end. Accounting for (17) at $\tilde\varphi_s = 1$, supplemented by a scalar-graviton potential $V_s(\sigma)$, and using (32) we now have in (38)

$$-\frac{\delta L_s}{\delta\sigma} \equiv W_s^V = \kappa_s^2 \nabla^\lambda\nabla_\lambda\sigma + \partial V_s/\partial\sigma \tag{39}$$

and

$$T_{s\mu\nu} = \kappa_s^2 \nabla_\mu\sigma\nabla_\nu\sigma - (\frac{1}{2}\kappa_s^2 g^{\kappa\lambda}\nabla_\kappa\sigma\nabla_\lambda\sigma - V_s)g_{\mu\nu}, \tag{40}$$

so that

$$\nabla^\nu T_{s\mu\nu} = W_s^V \partial_\mu\sigma. \tag{41}$$

Applying the covariant derivative to the first FE of (38) and using the truncated Bianchi identity in combination with the second FE of (38) we get that $\partial_\mu\lambda = 0$, so that

$$\lambda = \kappa_g^2\Lambda_0, \tag{42}$$

with Λ_0 being an arbitrary integration constant of the dimension mass squared. The scalar-graviton FE in (38) now looks like

$$W_s^U \equiv \kappa_s^2\nabla^\lambda\nabla_\lambda\sigma + \partial U_s/\partial\sigma = 0, \tag{43}$$

where U_s is the effective scalar-graviton potential

$$U_s = V_s + \kappa_g^2\Lambda_0 e^{-\sigma}. \tag{44}$$

Finally, the tensor-gravity FE reads

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda_g g_{\mu\nu} = \frac{1}{\kappa_g^2}(T_{s\mu\nu} + \Delta T_{s\mu\nu}), \tag{45}$$

with the conventional energy-momentum tensor $T_{s\mu\nu}$ acquiring (in compliance with (33)) the peculiar addition

$$\Delta T_{s\mu\nu} = -\delta L_s/\delta\sigma\, g_{\mu\nu} = \kappa_g^2\Lambda_0 e^{-\sigma} g_{\mu\nu}. \tag{46}$$

At that, the total energy-momentum tensor of the scalar graviton due to the scalar-graviton FE (43) is bound to be conserved (in compliance with the truncated Bianchi identity):

$$\nabla^\nu(T_{s\mu\nu} + \Delta T_{s\mu\nu}) = W_s^U\partial_\mu\sigma = 0. \tag{47}$$

At $\Lambda_0 = 0$, we clearly recover GR extended by the CC Λ_g and a conventional scalar field σ. At $\Lambda_0 > 0$ or $\Lambda_0 < 0$ we encounter two generic cases for the scalar graviton as a dark substance, respectively, DE [13] or DM [14]. For completeness, the last FE of (38) at $\Lambda_0 - \Lambda_\omega \neq 0$ with account for $e^{-\sigma} = \sqrt{-\omega}/\sqrt{-g}$ factorizes as[22]

$$\nabla_\kappa(e^{-\sigma}\Omega^{-1\kappa}{}_a) = \frac{1}{\sqrt{-g}}\partial_\kappa(\sqrt{-\omega}\Omega^{-1\kappa}{}_a) = 0, \tag{48}$$

where ω is to be extracted from g and σ due to the the previously derived tensor-gravity and scalar graviton FEs.[23] In a more general case, the solution for the Lagrange multiplier λ may depend on σ and the matter fields, what could make the nature of the emergent scalar-graviton dark substance in the quartet-modified gravity more contrived.

[22]In the limiting case $\Lambda_0 = \Lambda_\omega$ this relation may be adopted by default.

[23]Note that a previous study due to the author, in particular [14], concerning the matterless scalar-graviton GR extension with a non-dynamical scalar density remains, in fact, unchanged being added by a counterpart of (48) to treat the respective scalar density as dynamical.

5 Extending Weyl transverse gravity

5.1 Basic formalism

5.1.1 Generic case

Let now $\gamma = 1$ resulting in the effective metric $\tilde{g}_{\mu\nu} = \hat{g}_{\mu\nu}$ as follows

$$\hat{g}_{\mu\nu} \equiv e^{-\sigma/2} g_{\mu\nu} = (\omega/g)^{1/4} g_{\mu\nu}, \tag{49}$$

with an inverse $\hat{g}^{-1\mu\nu} = e^{\sigma/2} g^{\mu\nu} = (g/\omega)^{1/4} g^{\mu\nu}$. Such a metric is peculiar by the fact that $\hat{g} = \omega$ independent of the basic metric $g_{\mu\nu}$. Now we are putting for variations

$$\begin{aligned} \delta\hat{g}^{-1\kappa\lambda} &= e^{\sigma/2}(\delta^\kappa_\mu \delta^\lambda_\nu - \frac{1}{4}\hat{g}^{-1\kappa\lambda}\hat{g}_{\mu\nu})\delta g^{\mu\nu} - \frac{1}{4}\hat{g}^{-1\kappa\lambda}\omega^{-1\mu\nu}\delta\omega_{\mu\nu} \equiv \delta_T \hat{g}^{-1\kappa\lambda} + \delta_L \hat{g}^{-1\kappa\lambda}, \\ \delta\sqrt{-\hat{g}}/\sqrt{\hat{g}} &= -1/2\, \hat{g}_{\kappa\lambda}\delta\hat{g}^{-1\kappa\lambda} = 1/2\, \omega^{-1\mu\nu}\delta\omega_{\mu\nu} = \delta\sqrt{-\omega}/\sqrt{-\omega}, \end{aligned} \tag{50}$$

as well as making use of

$$\begin{aligned} \delta\sigma &= -1/2(e^{\sigma/2}\hat{g}_{\mu\nu}\delta g^{\mu\nu} + \omega^{-1\mu\nu}\delta\omega_{\mu\nu}), \\ \delta\sqrt{-g}/\sqrt{-g} &= -1/2\, e^{\sigma/2}\hat{g}_{\mu\nu}\delta g^{\mu\nu}. \end{aligned} \tag{51}$$

Adding to $\hat{\mathcal{L}}_{gsm} = \hat{L}_{gsm}\sqrt{-\hat{g}}$ the CC contribution $\Delta\mathcal{L}_\Lambda$ of eq. (7), equating to zero the coefficients at the variations of the total Lagrangian density $\hat{\mathcal{L}}_{\rm tot}$ with respect to the independent variations $\delta g^{\mu\nu}$, $\delta\omega_{\mu\nu}$ and $\delta\phi_I$, and separating the tensor-gravity equation onto the traceless/transversal and trace/longitudinal respective to $\hat{g}_{\mu\nu}$ parts we get similarly to the GR case the system of FEs as follows:

$$\begin{aligned} \hat{R}_{\mu\nu} - \frac{1}{4}\hat{R}\,\hat{g}_{\mu\nu} - \frac{1}{\kappa_g^2}(\hat{T}_{sm\mu\nu} - \frac{1}{4}\hat{T}_{sm}\,\hat{g}_{\mu\nu}) &= 0, \\ \frac{\delta\hat{L}_{sm}}{\delta\sigma} - \kappa_g^2\Lambda_g e^\sigma &= 0, \\ \hat{\nabla}_\mu\Big((\hat{R} + 4\Lambda_\omega + \frac{1}{\kappa_g^2}(\hat{T}_{sm} + 4\frac{\delta\hat{L}_{sm}}{\delta\sigma}))\Omega^{-1}{}^\mu_a\Big) &= 0, \\ \frac{\delta\hat{L}_{sm}}{\delta\phi_I} &= 0. \end{aligned} \tag{52}$$

In the above, we put canonically

$$\hat{T}_{sm\mu\nu} \equiv \frac{2}{\sqrt{-\hat{g}}}\frac{\partial(\sqrt{-\hat{g}}\hat{L}_{sm})}{\partial\hat{g}^{-1\mu\nu}} = 2\frac{\partial\hat{L}_{sm}}{\partial\hat{g}^{-1\mu\nu}} - \hat{L}_{sm}\hat{g}_{\mu\nu}, \tag{53}$$

with

$$\hat{T}_{sm} \equiv \hat{T}_{sm}{}^\lambda_\lambda = 2\frac{\partial\hat{L}_{sm}}{\partial\hat{g}^{-1\mu\nu}}\hat{g}^{-1\mu\nu} - 4\hat{L}_{sm}. \tag{54}$$

The FEs (52) (but for the third one) coincide with those for the WTDiff gravity extended by a specific (under $\Lambda_g \neq 0$) scalar field. In neglect (under the non-dynamical ω) by the third FE, the non-gravitating CC Λ_ω (in the line with the scalar curvature $\hat{R}$) drops out from FEs remaining completely irrelevant. Under the dynamical ω, Λ_ω drops out only in the formal limit $\Lambda_\omega \to \infty$, with the third FE factorizing in such a limit as $\partial_\kappa(\sqrt{-\omega}\Omega^{-1}{}^\kappa_a) = 0$ similarly to the GR case.

5.1.2 Energy-momentum constraint

In distinction with the GR extension, in the WTDiff case there appears a priori no matter covariant conservation condition (independent of the curvature). Namely, applying the covariant derivative to the first FE of (52) and accounting for the truncated Bianchi identity we get for the extended WTDiff gravity the constraint as follows:

$$\frac{1}{\kappa_g^2}\hat{\nabla}^\nu \hat{T}_{sm\mu\nu} = \frac{1}{4}\partial_\mu(\hat{R} + \frac{1}{\kappa_g^2}\hat{T}_{sm}), \tag{55}$$

with the explicit dependence on the curvature $\hat{R}$. So, the two extended theories are explicitly non-equivalent due to the different behavior of the covariant conservation/continuity condition. Still, for a particular choice of the Lagrangian $\hat{L}_{sm}$ (or a specific solution to FEs) which results in the fulfillment of such a condition

$$\hat{\nabla}^\nu \hat{T}_{sm\mu\nu} = 0 \tag{56}$$

there follows from (55) the constraint

$$\hat{R} + \frac{1}{\kappa_g^2}\hat{T}_{sm} = -4\Lambda_0, \tag{57}$$

with Λ_0 an arbitrary integration constant. Combining this restriction with the first FE of (52) we get the conventional tensor-gravity FE as follows:

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} - \Lambda_0 \hat{g}_{\mu\nu} = \frac{1}{\kappa_g^2}\hat{T}_{sm\mu\nu}, \tag{58}$$

with Λ_g entering only through the second FE of (52) for σ. At last, the third FE of (52) now reads

$$\hat{\nabla}_\mu\Big((\Lambda_\omega - \Lambda_0 + \Lambda_g e^\sigma)\Omega^{-1}{}^\mu_a\Big) = 0 \tag{59}$$

factorizing in the limit $\Lambda_\omega \to \infty$ as before: $\hat{\nabla}_\kappa \Omega^{-1}{}^\kappa_a = \partial_\kappa(\sqrt{-\omega}\Omega^{-1}{}^\kappa_a)/\sqrt{-\omega} = 0$. Clearly, even under the assumed covariant energy-momentum conservation the scalar-graviton extension to WTDiff is not completely equivalent to a similar extension to GR given by (31) (with the conservation of both $T_{sm\mu\nu}$ and $\Delta T_{sm\mu\nu}$). Generally though, the covariant conservation/continuity of the energy-momentum tensor in the WTDiff case is an additional requirement, not bound to be satisfied, what makes the equivalence of the extended theories even more problematic.[24]

5.2 Lagrange multiplier formalism

5.2.1 Generic case

Add now the Lagrangian density $\Delta\hat{\mathcal{L}}_\lambda$ given by (37) with a Lagrange multiplier $\hat{\lambda}$, and treat $\hat{\lambda}$ and σ as the independent field variables in addition to $g_{\mu\nu}$ and Ω^a. The variations of $\hat{g}^{-1\mu\nu} = e^{\sigma/2}g^{\mu\nu}$ and $\sqrt{-\hat{g}} = e^{-\sigma}\sqrt{-g}$ now look like

$$\begin{aligned} \delta\hat{g}^{-1\mu\nu} &= e^{\sigma/2}\delta g^{\mu\nu} + \frac{1}{2}\hat{g}^{-1\mu\nu}\delta\sigma, \\ \delta\sqrt{-\hat{g}}/\sqrt{-\hat{g}} = -1/2\,\hat{g}_{\mu\nu}\delta\hat{g}^{-1\mu\nu} &= -1/2\,e^{\sigma/2}\hat{g}_{\mu\nu}\delta g^{\mu\nu} - \delta\sigma, \end{aligned} \tag{60}$$

[24]Note nevertheless that at the non-dynamical ω and in the neglect by the scalar graviton σ the WTDiff gravity under the matter conservation gets classically equivalent to GR up to $\Lambda_0 \leftrightarrow \Lambda_g$ and $\hat{g}_{\mu\nu} \leftrightarrow g_{\mu\nu}$.

with $\delta\sqrt{-\omega}/\sqrt{-\omega}$ given by (28) and $\delta\sqrt{-g}/\sqrt{-g}$ by (51) as before. Extremizing the total action with respect to the independent variations $\delta\hat{\lambda}$, $\delta g^{\mu\nu}$, $\delta\sigma$ and $\delta\Omega^a$, we get first the constraint $e^\sigma = \sqrt{-g}/\sqrt{-\omega}$ and then FEs for $\hat{g}_{\mu\nu}$, σ and Ω^a, respectively, as follows

$$\begin{aligned} \hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} - \Lambda_g e^\sigma \hat{g}_{\mu\nu} &= \frac{1}{\kappa_g^2}(\hat{T}_{sm\mu\nu} - \hat{\lambda}\hat{g}_{\mu\nu}), \\ \hat{R} + \frac{1}{\kappa_g^2}(\hat{T}_{sm} + 4\frac{\delta\hat{L}_{sm}}{\delta\sigma} - 4\hat{\lambda}) &= 0, \\ \hat{\nabla}_\kappa((\hat{\lambda} + \kappa_g^2\Lambda_\omega)\Omega^{-1\kappa}{}_a) &= 0, \end{aligned} \tag{61}$$

with FE for matter remaining the same. Taking trace of the first FE above, combining the result with the second FE to excluding $\hat{\lambda}$, we get first $\delta\hat{L}_{sm}/\delta\sigma = \kappa_g^2\Lambda_g e^\sigma$ and then recover the previous λ-independent formalism.

5.2.2 Energy-momentum constraint

Applying the covariant derivative to the tensor-gravity FE above and accounting for the truncated Bianchi identity we get for the extended WTDiff gravity the constraint as follows:

$$\hat{\nabla}^\nu\hat{T}_{sm\mu\nu} = \partial_\mu(\hat{\lambda} - \kappa_g^2\Lambda_g e^\sigma). \tag{62}$$

In the case if $\hat{T}_{sm\mu\nu}$ satisfies the covariant conservation/continuity condition

$$\hat{\nabla}^\nu\hat{T}_{sm\mu\nu} = 0, \tag{63}$$

we get

$$\hat{\lambda} = \kappa_g^2(\Lambda_g e^\sigma - \Lambda_0), \tag{64}$$

with Λ_0 an arbitrary integration constant of the dimension mass squared. With the first FE of (61) becoming now

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} - \Lambda_0\hat{g}_{\mu\nu} = \frac{1}{\kappa_g^2}\hat{T}_{sm\mu\nu}, \tag{65}$$

the tensor gravity in this case reduces to the (conventional) GR with a CC Λ_0. Combining, in turn, this result with the second FE of (61) we get the (unconventional) scalar-graviton FE:

$$\delta\hat{L}_{sm}/\delta\sigma - \kappa_g^2\Lambda_g e^\sigma = 0. \tag{66}$$

The quartet FE clearly looks like (59). Stress ones again, that the covariant conservation (63) in the WTDiff extension is an additional assumption, not bound to be fulfilled.[25]

5.2.3 Matterless case

Apply the Lagrange multiplier formalism to the pure-gravity matterless case, where the consideration may be proceeded up to the end, with $\hat{L}_{sm} = \hat{L}_s$ given by (17) at $\tilde{\varphi}_s = \hat{\varphi}_s = 1$ supplemented by a scalar-graviton potential $\hat{V}_s(\sigma)$. First, it follows from (61), in compliance with the second FE of (52), that the wave operator

$$\hat{W}_s^V \equiv -\delta\hat{L}_s/\delta\sigma = \kappa_s^2\hat{\nabla}^\lambda\hat{\nabla}_\lambda\sigma + \partial\hat{V}_s/\partial\sigma \tag{67}$$

[25] Remind that a similar phenomenon, generally, takes place for the scalar-graviton GR/GDiff extension where the conventional energy-momentum conservation is restored just through the addition to $T_{sm\mu\nu}$ of the term $\Delta T_{sm\mu\nu}$ given by (33). Thus, the non-conservation of the conventional energy-momentum may more generally be a herald of the emergence of a non-conventional dark substance, here given by the scalar graviton.

satisfies the relation

$$\hat{W}_s^V = -\kappa_g^2 \Lambda_g e^\sigma. \tag{68}$$

By this token, the scalar-graviton FE now reads

$$\hat{W}_s^U \equiv \kappa_s^2 \hat{\nabla}^\lambda \hat{\nabla}_\lambda \sigma + \partial \hat{U}_s / \partial \sigma = 0, \tag{69}$$

with the effective scalar-graviton potential

$$\hat{U}_s \equiv \hat{V}_s + \kappa_g^2 \Lambda_g e^\sigma. \tag{70}$$

Likewise, we get that

$$\hat{T}_{s\mu\nu} = \kappa_s^2 \hat{\nabla}_\mu \sigma \hat{\nabla}_\nu \sigma - (\frac{1}{2}\kappa_s^2 \hat{g}^{-1\kappa\lambda} \hat{\nabla}_\kappa \sigma \hat{\nabla}_\lambda \sigma - \hat{V}_s) \hat{g}_{\mu\nu} \tag{71}$$

satisfies the relation

$$\hat{\nabla}^\nu \hat{T}_{s\mu\nu} = \hat{W}_s^V \partial_\mu \sigma, \tag{72}$$

implying the modified covariant conservation condition

$$\hat{\nabla}^\nu (\hat{T}_{s\mu\nu} + \Delta \hat{T}_{s\mu\nu}) = 0, \tag{73}$$

with

$$\Delta \hat{T}_{s\mu\nu} = \kappa_g^2 \Lambda_g e^\sigma \hat{g}_{\mu\nu}. \tag{74}$$

By this token, applying the truncated Bianchi identity to the first FE of (61) and integrating the result we get

$$\hat{\lambda} = -\kappa_g^2 \Lambda_0, \tag{75}$$

with Λ_0 an arbitrary integration constant. Henceforth, in addition to the scalar-graviton FE (69) we get the tensor gravity one as follows

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} - \Lambda_0 \hat{g}_{\mu\nu} = \frac{1}{\kappa_g^2}(\hat{T}_{s\mu\nu} + \Delta \hat{T}_{s\mu\nu}). \tag{76}$$

For completeness, the quartet FE of (61) at $\Lambda_0 - \Lambda_\omega \neq 0$ looks now like[26]

$$\hat{\nabla}_\kappa \Omega^{-1\kappa}{}_a = \frac{1}{\sqrt{-\omega}} \partial_\kappa (\sqrt{-\omega} \Omega^{-1\kappa}{}_a) = 0, \tag{77}$$

similarly to eq. (48) for the pure gravity extended GR.[27] At $\Lambda_g = 0$, in compliance with the ensuing covariant conservation of $\hat{T}_{s\mu\nu}$, we recover GR extended through a conventional scalar field σ and a spontaneous term Λ_0.

6 Summary: WTDiff vs. GR and beyond

The developed quartet-modified gravity possesses by the two generic traits: first, the emergence of a gravitational (particularly, the scalar-graviton) dark substance (as DM, DE, etc) and, second, the conceivable splitting of the vacuum energy and the respective CC onto the two parts of the different nature – the gravitating Λ_g and the non-gravitating Λ_ω – followed by a (partial) screening of them.

[26] At $\Lambda_0 = \Lambda_\omega$ this result may be adopted by continuity.

[27] Remind that in the cases at hand the canonical energy-momentum tensor $\hat{T}_{s\mu\nu}$ is not conserved on its own, henceforth the difference of eq. (77) from (59) where the canonical energy-momentum tensor was supposed to be conserved. The results though coincide at $\Lambda_g = \Lambda_0 = 0$.

What is most crucial, is that the large non-gravitating CC Λ_ω, being considered as a dominant vacuum energy contribution, drops out FEs in the limit $\Lambda_\omega \to \infty$, being thus suppressed at the large finite values. Such a phenomenon seems to be typical within the quartet-metric frameworks irrespective of the particular realization mode.

On the other hand, the behavior of a remaining gravitating part of the vacuum energy may depend significantly on a realization mode. The quartet-modified paradigm being extremely reach in its prospects for going beyond GR, possesses, by the same token, by many ambiguities. Besides the evident ambiguity in choosing the effective Lagrangian, there is an additional one in choosing an alternative prototype theory – GR or WTDiff – on which the extension should be built. The matter is that though the given prototype theories are classically equivalent their extensions ceases, generally, to be such. This is explicitly demonstrated in the case of the pure scalar reduction of the quartet-modified gravity in the two generic cases – GDiff and WGDiff – under the simplest choice of the scalar-graviton Lagrangian, where the consideration was executed up to the end.

More particularly, the GDiff gravity results in appearance of a scalar-graviton field satisfying the canonical FE (43) with a non-conventional potential (44), as well as the tensor-gravity FE (45) containing the specific addition (46) to the canonical scalar-field energy-momentum tensor. Both the tensor and scalar FEs contain a scalar-graviton dark substance dependent on an arbitrary integration constant Λ_0. At that, the gravitating CC Λ_g influences the tensor-gravity FE directly, signifying a residual CC problem as in GR itself.

On the other hand, in the WGDiff gravity Λ_g enters through a specific addition (74) to the canonical energy-momentum tensor of the scalar field. The latter satisfies the canonical FE (69), though with an unconventional effective potential $\hat{U}_s$ (70) dependent on the gravitating CC Λ_g as a source. On the contrary, the tensor-gravity FE (76) looks similarly to GR, but with the integration constant Λ_0 entering now as an emergent CC. This may ultimately indicate a way of the (partial) solving of the vacuum energy/CC problem in the quartet-modified gravity frameworks (at least, at a classical level).

Qualitatively, in the GDiff gravity the vacuum remains the bare one, with the tensor gravity described by the bare metric $g_{\mu\nu}$, while the matter "living" in the tensor-graviton vacuum modified by a scalar graviton substance. On the other hand, in the WGDiff gravity the scalar-graviton substance reforms the vacuum drastically, with the tensor gravity described by an effective metric $\hat{g}_{\mu\nu}(g_{\mu\nu}, \sigma)$ and the matter "living" in a scalar-tensor-graviton vacuum. In essence, the difference between the two alternatives lies in the kind of the gravitons – tensor or scalar – which "sweeps" from the vacuum proportionally to the gravitating CC Λ_g.

To conclude, when going beyond GR in the quartet-modified frameworks with the scalar graviton as an emergent dark substance, the dichotomy for a choice from the classically equivalent prototype theories – GR vs. WTDiff – resulting in the non-equivalent dynamically closed theories a priori should be taken into consideration at least on par. Hopefully, further studying and comparing the two routes may shed more light on the physics of the scalar graviton as an astroparticle and its bearing on cosmology.

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