Динамические системы с симметрией и редукция континуальных интегралов

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1. Введение

2. Геометрия конфигурационного пространства специальной механической системы с симметрией

3. Редукция континуального интеграла для рассматриваемой системы

4. Геометрическое представление якобиана редукции

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Dynamical systems with a symmetry

Infinite-dimensional: Yang-Mills theories,...

In $A_0 = 0$ gauge, $\mathcal{L}$ for interaction of Y-M field with a scalar field is

$$\mathcal{L} = -\frac{1}{2g_0^2} k_{\alpha\beta}(\partial_0 A_i^\alpha)(\partial_0 A^{i\beta}) + \frac{1}{2} G_{ab}(\partial_0 f^a)(\partial_0 f^b) - \frac{1}{4g_0^2} k_{\alpha\beta} F_{ij}^\alpha F^{\beta ij} + \frac{1}{2} G_{ab}(\nabla_i f^a)(\nabla^i f^b) - V_0(A, f).$$

$k_{\alpha\beta} = c^\tau_{\mu\alpha} c^\mu_{\tau\beta}$, V_0 is some gauge-invarint potential.

$$(\nabla f)_i^a(\bar{x}, t) = (\delta_b^a \partial_i(\bar{x}) - (\bar{J}_\alpha)^a_b A_i^\alpha(\bar{x}, t)) f^b(\bar{x}, t),$$

$\bar{J}_\alpha$ are the generators of the reps $\bar{D}^n_m(a)$ acting on the vector space V:
$\hat{f}^n = \bar{D}^n_m(a) f^m$, $\bar{D}^n_m(\Phi(g,h)) = \bar{D}^m_p(h)\bar{D}^p_n(g)$.
$[\bar{J}_\alpha, \bar{J}_\beta] = \bar{c}^\gamma_{\alpha\beta} \bar{J}_\gamma$ $\bar{c}^\gamma_{\alpha\beta} = -c^\gamma_{\alpha\beta}$.

The Lagrangian $\mathcal{L}$ is invariant under time-independent gauge transformations of the gauge potentials and scalar fields: :

$$\begin{aligned} \tilde{A}_i^{\alpha}(\mathbf{x},t) &= \rho_{\beta}^{\alpha}(g^{-1}(\mathbf{x}))A_i^{\beta}(\mathbf{x},t) + u_{\mu}^{\alpha}(g(\mathbf{x}))\frac{\partial g^{\mu}(\mathbf{x})}{\partial \mathbf{x}^i}\,, \\ \tilde{f}^a(\mathbf{x},t) &= \bar{D}_b^a(g(\mathbf{x}))f^b(\mathbf{x},t). \end{aligned}$$

The obtained Lagrangian looks as if it represents the motion of two "particle" in the product space $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$ in the invariant potential

$$V[A,f] = \int d^3x\Big[\frac{1}{2}\,k_{\alpha\beta}\,F_{ij}^{\alpha}(\mathbf{x})\,F^{\beta\;ij}(\mathbf{x}) - \frac{1}{2}G_{ab}(\nabla f)_i^a(\mathbf{x})(\nabla f)^{b\,i}(\mathbf{x}) + V_0\Big].$$

One of the space, $\mathcal{P}$, is an infinite-dimensional Riemannian manifold. The gauge fields A_i^a can be regarded as points of this manifold. And the other space, $\mathcal{V}$, is the space of functions with the values in the vector space $\mathcal{V}$. Also, we are given an action of the group, the group of the gauge transformations, on the product space.

This is analogous to what we have in reduction problem for dynamical system with symmetry in mechanics where we are interested in description of an internal dynamics given on the orbit space.

The mechanical system with a symmetry on the manifold $\mathcal{P} \times \mathcal{V}$

The mechanical system describes the motion of two interacting scalar particles on the product manifold $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$, where $\mathcal{P}$ is a Riemannian manifold and $\mathcal{V}$ is a finite-dimensional vector space.
(Q^A, f^n), $A = 1, \ldots, N_{\mathcal{P}}$ and $n = 1, \ldots, N_{\mathcal{V}}$ are the local coordinates of a point $(p, v) \in \tilde{\mathcal{P}}$
The Riemannian metric on $\tilde{\mathcal{P}}$:

$$ds^2 = G_{AB}(Q) dQ^A dQ^B + G_{mn} df^m df^n, \quad G_{mn} \ -\text{const}$$

There is a free proper isometric action of a compact semi-simple Lie group $\mathcal{G}$ on $\tilde{\mathcal{P}}$: $(p, v)g = (pg, g^{-1} v)$:

$$\tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^n = \bar{D}^n_m(g) f^m.$$

$\bar{D}^n_m(g) \equiv D^n_m(g^{-1})$. The Lagrangian

$$\mathcal{L} = \frac{1}{2} G_{AB}(Q)\, \dot{Q}^A \dot{Q}^B + \frac{1}{2} G_{mn}\, \dot{f}^m \dot{f}^n - V(Q, f).$$

Assuming that $V(Q, f) = V(F(Q, a), \bar{D}(a) f) \Rightarrow \mathcal{L}$ is also invariant.

Isometry

Due to isometry, there are two relations:

$$G_{AB}(Q) = G_{DC}(F(Q,g))F^D_A(Q,g)F^C_B(Q,g),$$

with $F^B_A(Q,g) \equiv \frac{\partial F^B(Q,g))}{\partial Q^A}$ and

$$G_{pq} = G_{mn}\bar{D}^m_p(g)\bar{D}^n_q(g),$$

The Killing vector fields are defined as

$K^A_\alpha(Q)\frac{\partial}{\partial Q^A}$ with $K^A_\alpha(Q) = \frac{\partial \tilde{Q}^A}{\partial a^\alpha}\Big|_{a=e}$;

$K^n_\alpha(f)\frac{\partial}{\partial f^n}$ with $K^n_\alpha(f) = \frac{\partial \tilde{f}^n}{\partial a^\alpha}\Big|_{a=e} = \frac{\partial \bar{D}^n_m(a)}{\partial a^\alpha}\Big|_{a=e} = (\bar{J}_\alpha)^n_m f^m.$

Principal fiber bundle $\pi' : \mathcal{P} \times \mathcal{V} \to \mathcal{P} \times_{\mathcal{G}} \mathcal{V}$

$\pi' : (p, v) \to [p, v]$, $[p, v]$ is the equivalence class: $(p, v) \sim (pg, g^{-1}v)$.

$\tilde{\mathcal{P}}$ – a total space of this principal fiber bundle.

$\tilde{\mathcal{M}} = \mathcal{P} \times_{\mathcal{G}} \mathcal{V}$ – the orbit space manifold, the base space of π'.

How to find the bundle coordinates for $(p, v) \in \tilde{\mathcal{P}}$?

the section $\tilde{\sigma}_i$ of $\mathrm{P}(\tilde{\mathcal{M}}, \mathcal{G})$, $\pi' \cdot \tilde{\sigma}_i = \mathrm{id}$:

$$\tilde{\sigma}_i([p, v]) = (\sigma_i(x), a(p)v),$$

σ_i is the local section of $\mathrm{P}(\mathcal{M}, \mathcal{G})$,
$a(p)$ is the group element such that $p = \sigma_i(x)a(p)$, $p \in \mathcal{P}$.

Since $(\sigma_i(x), a(p)\, v) = (p\, a^{-1}(p), a(p)\, v) = (p, v)\, a^{-1}(p)$,

$$\tilde{\sigma}_i([p, v]) = (p, v)\, a^{-1}(p).$$

The gauge surfaces $\Sigma_i \in \mathcal{P}$ and bundle coordinates in $\mathrm{P}(\mathcal{M},\mathcal{G})$

$\sigma_i : U_i \to \Sigma_i \in \pi_{\mathrm{P}}^{-1}(U_i)$, $x = \pi_{\mathrm{P}}(p)$

The local submanifold Σ is given by $\{\chi^\alpha(Q) = 0,\ \alpha = 1, ..., n^{\mathcal{G}}\}$.

($p \in \Sigma$ if its coordinates Q^{*A}: $\{\chi^\alpha(Q^{*A}) = 0\}$.)

The bundle coordinates of p are $(Q^{*A}(Q), a^\alpha(Q))$.

The group coordinates $a^\alpha(Q)$ are defined by the solution of the following equation:

$$\chi^\alpha(F^A(Q, a^{-1}(Q))) = 0.$$

(This group element carries the point p to the submanifold Σ.)
The coordinates $Q^{*A}(Q)$ are from

$$Q^{*A} = F^A(Q, a^{-1}(Q))).$$

The coordinates $Q^{*A}(Q)$ – dependent coordinates.

Resolved gauges

If the local submanifold Σ can be defined parametrically:

$Q^A = Q^{*A}(x^i), \qquad \chi^\alpha(Q^{*A}(x^i)) = 0.$

The coordinates of p are $(x^i(Q), a^\alpha(Q))$. $x^i(Q)$ are determined by the equation

$$Q^{*A}(x^i) = F^A(Q, a^{-1}(Q))).$$

Invariant coordinates x^i can be identified with the coordinates given on the base manifold $\mathcal{M}$.

The adapted coordinates in the principal bundle $\pi' : \mathcal{P} \times \mathcal{V} \to \mathcal{P} \times_{\mathcal{G}} \mathcal{V}$.

In $\tilde{\mathcal{P}}$, the point (p, v) have the coordinates (Q^A, f^b).

If $(p, v) \in \tilde{\Sigma}$ then its cordinates are $(Q^{*A}, \tilde{f}^a)$, (or $(Q^{*A}(x), \tilde{f}^a)$ in resolved gauge case).

The coordinate functions in the bundle are

$$\tilde{\varphi}_i^{-1} : (Q^A, f^b) \to (Q^{*A}(Q), \tilde{f}^b(Q), a^\alpha(Q)\,),$$

(or $\to (x^i(Q), \tilde{f}^b(Q), a^\alpha(Q)\,)$,

$$\tilde{f}^b(Q) = D^b_c(a(Q))\, f^c,$$

$(\bar{D}^b_c(a^{-1}) \equiv D^b_c(a))$.
The coordinate function $\tilde{\varphi}_i$

$$\tilde{\varphi}_i : (Q^{*B}, \tilde{f}^b, a^\alpha) \to (F^A(Q^*, a), \bar{D}^c_b(a)\tilde{f}^b).$$

The replacement of the coordinates (Q^A, f^b) of a point (p, v):

$$Q^A = F^A(Q^*(x^i), a^\alpha), \quad f^b = \bar{D}^b_c(a)\tilde{f}^c, \quad a^\alpha = a^\alpha(Q)$$

The metric on $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$ in the basis $\{\partial/\partial x^i, \partial/\partial \tilde{f}^b, \partial/\partial a^\alpha\}$

$$G_{\tilde{A}\tilde{B}} = \left(\begin{array}{ccc} \tilde{h}_{ij} + \mathcal{A}_i^\mu \mathcal{A}_j^\nu d_{\mu\nu} & 0 & \mathcal{A}_i^\mu d_{\mu\nu} \bar{u}_\alpha^\nu(a) \\ 0 & G_{ab} & \mathcal{A}_a^\mu d_{\mu\nu} \bar{u}_\alpha^\nu(a) \\ \mathcal{A}_j^\mu d_{\mu\nu} \bar{u}_\beta^\nu(a) & \mathcal{A}_b^\mu d_{\mu\nu} \bar{u}_\beta^\nu(a) & d_{\mu\nu} \bar{u}_\alpha^\mu(a) \bar{u}_\beta^\nu(a) \end{array} \right),$$

where $(\mathcal{A}_i^\alpha, \mathcal{A}_p^\alpha)$ are the components of the mechanical connection in the principal fiber bundle $\mathrm{P}(\tilde{\mathcal{M}}, \mathcal{G})$:

$$\mathcal{A}_i^\alpha(x, \tilde{f}) = d^{\alpha\beta} K_\beta^C G_{DC} Q^{\star}{}_i^D, \quad \mathcal{A}_p^\alpha(x, \tilde{f}) = d^{\alpha\beta} K_\beta^r G_{rp}.$$

$\tilde{h}_{ij}(x, \tilde{f}) = Q^{*}{}_i^A \tilde{G}^{\mathrm{H}}_{AB} Q^{*}{}_j^B$, and $\tilde{G}^{\mathrm{H}}_{AB} = G_{AB} - G_{AC} K_\mu^C d^{\mu\nu} K_\nu^D G_{DB}$.

$$d_{\mu\nu}(x, \tilde{f}) = \gamma_{\mu\nu}(x) + \gamma'(\tilde{f}) = K_\mu^A G_{AB} K_\nu^B + K_\mu^r G_{rp} K_\nu^p$$

$G^{\tilde{A}\tilde{B}}$

$$G^{\tilde{A}\tilde{B}} = \begin{pmatrix} h^{ij} & \underset{(\gamma)}{\mathcal{A}}{}^{\mu}_{m} K^{a}_{\mu} h^{mj} & -h^{nj}\, \underset{(\gamma)}{\mathcal{A}}{}^{\beta}_{n} \bar{v}^{\alpha}_{\beta} \\ \underset{(\gamma)}{\mathcal{A}}{}^{\mu}_{m} K^{b}_{\mu} h^{ni} & G^{AB} N^{a}_{A} N^{b}_{B} + G^{ab} & -G^{EC} \Lambda^{\beta}_{E} \Lambda^{\mu}_{C} K^{b}_{\mu} \bar{v}^{\alpha}_{\beta} \\ -h^{ki} \underset{(\gamma)}{\mathcal{A}}{}^{\varepsilon}_{k} \bar{v}^{\beta}_{\varepsilon} & -G^{EC} \Lambda^{\varepsilon}_{E} \Lambda^{\mu}_{C} K^{a}_{\mu} \bar{v}^{\beta}_{\varepsilon} & G^{BC} \Lambda^{\alpha'}_{B} \Lambda^{\beta'}_{C}\, \bar{v}^{\alpha}_{\alpha'} v^{\beta}_{\beta'} \end{pmatrix}.$$

Here $\Lambda^{\beta}_{E} = (\Phi^{-1})^{\beta}_{\mu} \chi^{\mu}_{E}$, h^{ij} is an inverse matrix to the matrix

$h_{ij} = Q^{*A}_{i} G^{\rm H}_{AB} Q^{*B}_{j}$ with $G^{\rm H}_{AB} = G_{AB} - G_{AD} K^{D}_{\alpha} \gamma^{\alpha\beta} K^{C}_{\beta} G_{CB}$.

$$\underset{(\gamma)}{\mathcal{A}}{}^{\mu}_{m} = \gamma^{\mu\nu} K^{A}_{\nu} G_{AB} Q^{*B}_{m}.$$

χ^{α}_{B} we denote $\chi^{\alpha}_{B} = \partial \chi^{\alpha}(Q)/\partial Q^{B}|_{Q=Q^{*}(x)}$, $(\Phi)^{\alpha}_{\beta} = K^{A}_{\beta} \chi^{\alpha}_{A}$ is the Faddeev-Popov matrix, $N^{b}_{B} = -K^{b}_{\mu} (\Phi)^{\mu}_{\nu} \chi^{\nu}_{B} \equiv -K^{b}_{\mu} \Lambda^{\mu}_{B}$ is one of the components of a particular projector
$N = (N^{A}_{B} = \delta^{A}_{B} - K^{A}_{\mu} \Lambda^{\mu}_{B}, N^{b}_{B}, N^{B}_{b} = 0, N^{a}_{b} = \delta^{a}_{b})$ onto a tangent space to the orbit space $\tilde{\mathcal{M}}$.

det

$$\det G_{\tilde{A}\tilde{B}} = (\det d_{\alpha\beta})\,(\det \bar{u}^{\mu}_{\nu}(a))^2 \det \begin{pmatrix} \tilde{h}_{ij} & \tilde{G}^{\mathrm{H}}_{Bb} Q^{*B}_{i} \\ \tilde{G}^{\mathrm{H}}_{Aa} Q^{*A}_{j} & \tilde{G}^{\mathrm{H}}_{ba} \end{pmatrix}, \qquad (1)$$

where $\tilde{G}^{\mathrm{H}}_{Aa} = -G_{AB} K^{B}_{\mu} d^{\mu\nu} K^{b}_{\nu} G_{ba}$, $\tilde{G}^{\mathrm{H}}_{ba} = G_{ba} - G_{bc} K^{c}_{\mu} d^{\mu\nu} K^{p}_{\nu} G_{pa}$.
The determinant on the RHS of (1) is the determinant of the metric defined on the orbit space $\tilde{\mathcal{M}} = \mathcal{P} \times_{\mathcal{G}} \mathcal{V}$. It will be denoted by H.

The backward Kolmogorov equation on $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$

$$\begin{cases} \left(\frac{\partial}{\partial t_a} + \frac{1}{2}\mu^2\kappa\left[\triangle_{\mathcal{P}}(p_a) + \triangle_{\mathcal{V}}(v_a)\right] + \frac{1}{\mu^2\kappa m}V(p_a, v_a)\right)\psi_{t_b}(p_a, t_a) = 0, \\ \psi_{t_b}(p_b, v_b, t_b) = \phi_0(p_b, v_b), \qquad\qquad (t_b > t_a), \end{cases}$$

$\mu^2 = \frac{\hbar}{m}$, κ is a real positive parameter, $V(pg, g^{-1}v) = V(p, v)$, $g \in \mathcal{G}$,

$$\triangle_{\mathcal{P}}(Q) = G^{-1/2}(Q)\frac{\partial}{\partial Q^A}G^{AB}(Q)G^{1/2}(Q)\frac{\partial}{\partial Q^B}, \quad G = det(G_{AB})$$

$$\triangle_{\mathcal{V}}(f) = G^{ab}\frac{\partial}{\partial f^a \partial f^b},$$

$$\psi_{t_b}(p_a, v_a, t_a) = \mathrm{E}\Big[\phi_0(\eta_1(t_b), \eta_2(t_b)) \exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} V(\eta_1(u), \eta_2(u))du\}\Big]$$

$$= \int_{\Omega_-} d\mu^{\eta}(\omega)\phi_0(\eta(t_b)) \exp\{\ldots\},$$

$\eta(t) = (\eta_1(t), \eta_2(t))$ is a global stochastic process on $\tilde{\mathcal{P}}$,
μ^{η} – p.i. measure on paths $\Omega_- = \{\omega(t) = \omega^1(t) \times \omega^2(t) : \omega^{1,2}(t_a) = 0, \eta_1(t) = p_a + \omega^1(t), \eta_2(t) = v_a + \omega^2(t)\}$
SDE in chart $(U_{\mathcal{P}} \times U_{\mathcal{V}}, \varphi)$:

$$d\eta_1^A(t) = \frac{1}{2}\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})dt + \mu\sqrt{\kappa}\mathcal{X}^A_{\bar{M}}(\eta_1(t))dw^{\bar{M}}(t),$$

$$d\eta_2^a(t) = \mu\sqrt{\kappa}\mathcal{X}^b_{\bar{a}}dw^{\bar{b}}(t)$$

$(\sum_{\bar{K}=1}^{n_{\mathcal{P}}}\mathcal{X}^A_{\bar{K}}\mathcal{X}^B_{\bar{K}} = G^{AB}$ and $\sum_{\bar{a}=1}^{n_{\mathcal{V}}}\mathcal{X}^b_{\bar{a}}\mathcal{X}^c_{\bar{a}} = G^{bc})$,
$dw^{\bar{M}}(t)$ and $dw^{\bar{b}}(t)$ – independent Wiener processes.

The path integral as the superposition of the local semigroups

$$\psi_{t_b}(p_a, v_a, t_a) = U(t_b, t_a)\phi_0(p_a, v_a) = \lim_q \tilde{U}_\eta(t_a, t_1) \cdot \ldots \cdot \tilde{U}_\eta(t_{n-1}, t_b)\phi_0(p_a, v_a),$$

the local semigroup $\tilde{U}_\eta$:

$$\tilde{U}_\eta(s,t)\phi(p,v) = \mathrm{E}_{s,p,v}\phi(\eta_1(t), \eta_2(t)) \quad s \le t \quad \eta_1(s) = p,\ \eta_2(s) = v.$$

The p.i. measures for $\tilde{U}_\eta$ is determined by the local processes $\varphi^{\tilde{\mathcal{P}}}(\eta_t) \equiv \{\eta_1^A(t), \eta_2^a(t)\}$.

Stochastic process corresponding to the bundle coordinates on $\tilde{\mathcal{P}}$

$\eta(t) \to \zeta(t)$ – the phase space transformation of the stochastic processes

$$\eta_1^A(t) = F^A(Q^{*B}(x^i(t)), a^\alpha(t)), \;\; \eta_2^a(t) = \bar{D}_c^a(a(t))\tilde{f}^c(t).$$

New local processes $\zeta^{\tilde{\varphi}^{\mathcal{P}}}(t) = (x^i(t), \tilde{f}^a(t), a^\alpha(t))$ define the measures in local semigroup:

$$\tilde{U}_\eta(s,t)\phi_0(p) = \mathrm{E}_{s,\tilde{\varphi}^{\tilde{\mathcal{P}}}(p,v)}\phi_0\left((\tilde{\varphi}^{\tilde{\mathcal{P}}})^{-1}(\zeta^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}(t))\right) = \mathrm{E}_{s,\tilde{\varphi}^{\tilde{\mathcal{P}}}(p,v)}\tilde{\phi}_0\left(\zeta^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}(t)\right),$$

The global semigroup for the process $\tilde{\zeta}(t)$ based on local $\tilde{\zeta}^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}(t)$:

$$\psi_{t_b}(p_a, v_a, t_a) = \lim_q \tilde{U}_{\tilde{\zeta}^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}}(t_a, t_1) \cdot \dots \cdot \tilde{U}_{\tilde{\zeta}^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}}(t_{n-1}, t_b)\tilde{\phi}_0(x_a, \tilde{f}_a, \theta_a),$$

$$\tilde{U}_{\tilde{\zeta}^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}}(s,t)\tilde{\phi}(x_0, \tilde{f}_0, \theta_0) = \mathrm{E}_{s,(x_0,\tilde{f}_0,\theta_0)}\tilde{\phi}(x(t), \tilde{f}(t), a(t)),$$

with $x(s) = x_0$, $\tilde{f}(s) = \tilde{f}_0$, $a(s) = \theta_0$.

Stochastic differential equations for $\tilde{\zeta}^{\tilde{\varphi}^{\tilde{\mathcal{P}}}}(t)$

$$
\begin{aligned}
dx_t^i &= \frac{1}{2}(\mu^2\kappa)b^i dt + \mu\sqrt{\kappa}\tilde{X}^i_{\bar{m}}\, d\tilde{w}_t^{\bar{m}} \\
d\tilde{f}_t^a &= \frac{1}{2}(\mu^2\kappa)b^a dt + \mu\sqrt{\kappa}(\tilde{X}^a_{\bar{m}}\, d\tilde{w}_t^{\bar{m}} + \tilde{X}^a_{\bar{b}}\, d\tilde{w}_t^{\bar{b}}) \\
da_t^\alpha &= \frac{1}{2}(\mu^2\kappa)b^\alpha dt + \mu\sqrt{\kappa}(\tilde{X}^\alpha_{\bar{m}}\, d\tilde{w}_t^{\bar{m}} + \tilde{X}^\alpha_{\bar{\beta}}\, d\tilde{w}_t^{\bar{\beta}} + \tilde{X}^\alpha_{\bar{b}}\, d\tilde{w}_t^{\bar{b}})
\end{aligned}
$$

$\tilde{w}_t^{\bar{m}}$, $\tilde{w}_t^{\bar{b}}$, $\tilde{w}_t^{\bar{\beta}}$ are independent. Diffusion coefficients:

1. $\tilde{X}^i_{\bar{m}}\tilde{X}^j_{\bar{m}} = h^{ij}$,
2. $\tilde{X}^i_{\bar{m}}\tilde{X}^a_{\bar{m}} = \underset{(\gamma)}{\mathcal{A}}{}^\mu_m K^a_\mu h^{mi}$,
3. $\tilde{X}^a_{\bar{m}}\tilde{X}^b_{\bar{m}} + \tilde{X}^a_{\bar{b}}\tilde{X}^b_{\bar{b}} = G^{AB}N^a_A N^b_B + G^{ab}$,
4. $\tilde{X}^i_{\bar{m}}\tilde{X}^\alpha_{\bar{m}} = -h^{ni}\underset{(\gamma)}{\mathcal{A}}{}^\beta_n \bar{v}^\alpha_\beta$,
5. $\tilde{X}^a_{\bar{m}}\tilde{X}^\alpha_{\bar{m}} + \tilde{X}^a_{\bar{b}}\tilde{X}^\alpha_{\bar{b}} = -\left(\gamma^{\mu\nu} + h^{ij}\underset{(\gamma)}{\mathcal{A}}{}^\mu_i \underset{(\gamma)}{\mathcal{A}}{}^\nu_j\right)K^a_\mu \bar{v}^\alpha_\nu$,

Factorization of the path integral measure

For the Markov processes

$$\tilde{U}_{\tilde{\zeta}\tilde{\varphi}^{\tilde{\mathcal{P}}}}(s,t)\tilde{\phi}(x_0,\tilde{f}_0,\theta_0) = \mathrm{E}\Big[\mathrm{E}\Big[\tilde{\phi}(x(t),\tilde{f}(t),a(t)) \mid (\mathcal{F}_{(x,\tilde{f})})^t_s\Big]\Big],$$

$$\hat{\tilde{\phi}}(x(t),\tilde{f}(t)) := \mathrm{E}\Big[\tilde{\phi}(x(t),\tilde{f}(t),a(t)) \mid (\mathcal{F}_{(x,\tilde{f})})^t_s\Big].$$

In the nonlinear filtering theory $\hat{f}(t) = \mathrm{E}[\mathrm{f}(\mathrm{Z},\mathrm{t})|\mathrm{Y}^{\mathrm{t}}_{\mathrm{t_0}}]$

$$\begin{aligned} d\hat{f}(t) &= \mathrm{E}[\mathrm{f_t} + \mathrm{f_z}\varphi + \frac{1}{2}\mathrm{f_{zz}}(\mathrm{XX^T})|\mathrm{Y}^{\mathrm{t}}_{\mathrm{t_0}}]\mathrm{dt} \\ &+\mathrm{E}\Big[\mathrm{f}(\varphi_1 - \hat{\varphi}_1) + \mathrm{f_z}(\mathrm{XX_1^T})|\mathrm{Y}^{\mathrm{t}}_{\mathrm{t_0}}\Big](\mathrm{X_1X_1^T})^{-1}(\mathrm{dY} - \hat{\varphi}_1\mathrm{dt}) \end{aligned}$$

$dY_t = \varphi_1 dt + X_1 dw_t$ and $dZ_t = \varphi dt + X dw_t$, $\hat{\varphi}_1 = \mathrm{E}\Big[\varphi_1(\mathrm{Y_t},\mathrm{Z_t},\mathrm{t})|\mathrm{Y_t}\Big]$

$$Y_t \Rightarrow \begin{pmatrix} dx^i_t \\ d\tilde{f}^a \end{pmatrix} = \frac{1}{2}(\mu^2\kappa)\begin{pmatrix} b^i \\ b^a \end{pmatrix}dt + \mu\sqrt{\kappa}\begin{pmatrix} \tilde{X}^i_{\bar{m}} & 0 \\ \tilde{X}^a_{\bar{m}} & \tilde{X}^a_{\bar{b}} \end{pmatrix}\begin{pmatrix} d\tilde{w}^{\bar{m}}_t \\ d\tilde{w}^{\bar{b}}_t \end{pmatrix}.$$

A signal process – $Z_t \Rightarrow a^\alpha_t$, the observable process – Y_t

$\tilde{\phi}(x, \tilde{f}, a) = \sum_{\lambda,p,q} c^{\lambda}_{pq}(x, \tilde{f}) D^{\lambda}_{pq}(a)$, $\Rightarrow$ the linear matrix equation:

$$d\hat{D}^{\lambda}_{pq} = \Gamma^{\mu}_{1}\ (J_{\mu})^{\lambda}_{pq'}\hat{D}^{\lambda}_{q'q}dt + \Gamma^{\mu\nu}_{2}\ (J_{\mu})^{\lambda}_{pq'}(J_{\nu})^{\lambda}_{q'q''}\hat{D}^{\lambda}_{q''q}dt$$

$$-(J_{\mu})^{\lambda}_{pq'}\hat{D}^{\lambda}_{q'q}\left(\underset{(\gamma)}{\mathcal{A}^{\mu}_{n}}\tilde{X}^{n}_{\bar{m}}d\tilde{w}^{\bar{m}}_{t} + \tilde{\mathcal{A}}^{\mu}_{a}\tilde{X}^{a}_{\bar{b}}d\tilde{w}^{\bar{b}}_{t}\right) \text{ with solution :}$$

$$\hat{D}^{\lambda}_{pq}(x(t), \tilde{f}(t)) = (\overleftarrow{\exp})^{\lambda}_{pn}(x(t), \tilde{f}(t), t, s)\,\mathrm{E}\big[D^{\lambda}_{nq}(a(s)) \mid (\mathcal{F}_{x,\tilde{f}})^{t}_{s}\big],$$

$$(\overleftarrow{\exp})^{\lambda}_{pn}(x(t), \tilde{f}(t), t, s) = \overleftarrow{\exp}\int_{s}^{t}\Big\{\mu^{2}\kappa\Big[\frac{1}{2}d^{\alpha\nu}(x(u), \tilde{f}(u))(J_{\alpha})^{\lambda}_{pr}(J_{\nu})^{\lambda}_{rn}$$

$$-\frac{1}{2}\frac{1}{\sqrt{d\,H}}\frac{\partial}{\partial x^{k}}\left(\sqrt{d\,H}h^{km}\underset{(\gamma)}{\mathcal{A}^{\nu}_{m}}(x(u))\right)(J_{\nu})^{\lambda}_{pn}$$

$$-\frac{1}{2}(G^{EC}\Lambda^{\nu}_{E}\Lambda^{\mu}_{C})\frac{1}{\sqrt{d\,H}}\frac{\partial}{\partial\tilde{f}^{b}}\left(\sqrt{d\,H}K^{b}_{\mu}\right)(J_{\nu})^{\lambda}_{pn}\Big]du$$

$$-\mu\sqrt{\kappa}\Big[\underset{(\gamma)}{\mathcal{A}^{\nu}_{k}}(x(u))\tilde{X}^{k}_{\bar{m}}(u)(J_{\nu})^{\lambda}_{pn}d\tilde{w}^{\bar{m}}(u) + \tilde{\mathcal{A}}^{\nu}_{a}\tilde{X}^{a}_{\bar{b}}(u)(J_{\nu})^{\lambda}_{pn}d\tilde{w}^{\bar{b}}(u)\Big]\Big\}$$

$(H, \tilde{\mathcal{A}}^{\nu}_{a}$ depend on $x(u)$ and $\tilde{f}(u))$

is a multiplicative stochastic integral (arrow: $t_b \Leftarrow t_a$)

The local semigroup:

$$\tilde{U}_{\tilde{\zeta}\tilde{\varphi}^{\tilde{\mathcal{P}}}}(s,t)\tilde{\phi}(x_0,\tilde{f}_0,\theta_0)=\sum_{\lambda,p,q,q'}\mathrm{E}\big[c^{\lambda}_{pq}(x(t),\tilde{f}(t))(\overleftarrow{\exp})^{\lambda}_{pq'}(x(t),t,s)\big]D^{\lambda}_{q'q}(\theta_0),$$

The global semigroup (symbolically):

$$\psi_{t_b}(p_a,v_a,t_a) = \sum_{\lambda,p,q,q'}\mathrm{E}\big[c^{\lambda}_{pq}(\xi(t_b))(\overleftarrow{\exp})^{\lambda}_{pq'}(\xi(t),t_b,t_a)\big]D^{\lambda}_{q'q}(\theta_a),$$

$$(\xi(t_a)=\pi'\circ(p_a,v_a)),$$

$\xi(t)=(\xi_1(t),\xi_2(t))$ is defined on the manifold $\tilde{\mathcal{M}}=\mathcal{P}\times_{\mathcal{G}}\mathcal{V}$. But

$$\psi_{t_b}(p_a,v_a,t_a) = \mathrm{E}\Big[\phi_0(\eta_1(t_b),\eta_2(t_b))\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\eta_1(u),\eta_2(u))du\}\Big]$$

$$= \int_{\Omega_-}d\mu^{\eta}(\omega)\phi_0(\eta(t_b))\exp\{\ldots\},$$

So, we get the relation between the path integrals: on $\tilde{\mathcal{P}}$ and on $\tilde{\mathcal{M}}$.

The differential generator (the Hamilton operator)

$$
\begin{aligned}
&\frac{1}{2}\mu^2\kappa\Big\{\Big[\triangle_{\tilde{M}} + h^{ni}\Big(\frac{1}{\sqrt{d}}\frac{\partial(\sqrt{d}\,)}{\partial x^n} + \underset{(\gamma)}{\mathcal{A}^{\nu}_{n}}\frac{1}{\sqrt{d}}\frac{\partial(\sqrt{d}K^b_{\nu})}{\partial \tilde{f}^b}\Big)\frac{\partial}{\partial x^i} \\
&+\Big(h^{mi}\underset{(\gamma)}{\mathcal{A}^{\mu}_{m}}K^a_{\mu}\frac{1}{\sqrt{d}}\frac{\partial(\sqrt{d}\,)}{\partial x^i} + (G^{ab}+G^{AB}N^a_A N^b_B)\frac{1}{\sqrt{d}}\frac{\partial(\sqrt{d}\,)}{\partial \tilde{f}^b}\Big)\frac{\partial}{\partial \tilde{f}^a}\Big](I^{\lambda})_{pq} \\
&-2h^{ni}\underset{(\gamma)}{\mathcal{A}^{\beta}_{n}}(J_{\beta})^{\lambda}_{pq}\frac{\partial}{\partial x^i} - 2h^{nk}\underset{(\gamma)}{\mathcal{A}^{\beta}_{n}}\underset{(\gamma)}{\mathcal{A}^{\mu}_{k}}K^a_{\mu}(J_{\beta})^{\lambda}_{pq}\frac{\partial}{\partial \tilde{f}^a} \\
&-2(\gamma^{\alpha\beta}K^a_{\alpha}K^b_{\beta} + G^{ab})\tilde{\mathcal{A}}^{\beta}_{a}(J_{\beta})^{\lambda}_{pq}\frac{\partial}{\partial \tilde{f}^b} \\
&-\Big[\frac{1}{\sqrt{dH}}\frac{\partial}{\partial x^i}\Big(\sqrt{dH}h^{ni}\underset{(\gamma)}{\mathcal{A}^{\beta}_{n}}\Big) + (G^{EC}\Lambda^{\beta}_{E}\Lambda^{\mu}_{C})\frac{1}{\sqrt{dH}}\frac{\partial(\sqrt{dH}K^b_{\mu})}{\partial \tilde{f}^b}\Big](J_{\beta})^{\lambda}_{pq} \\
&+(\gamma^{\alpha\beta} + h^{ij}\underset{(\gamma)}{\mathcal{A}^{\alpha}_{i}}\underset{(\gamma)}{\mathcal{A}^{\beta}_{j}})(J_{\alpha})^{\lambda}_{pq'}(J_{\beta})^{\lambda}_{q'q}\Big\}\ \text{acts on}\ \psi_{\mathrm{n}},
\end{aligned}
$$

$$
(\psi_n,\psi_m) = \int_{\tilde{\mathcal{M}}}\langle\psi_n,\psi_m\rangle_{V^*_{\lambda}}\sqrt{d(x,\tilde{f})}\,dv_{\tilde{\mathcal{M}}}(x,\tilde{f}),\ \psi_n\in\Gamma(\tilde{\mathcal{M}},V^*)
$$

The integral relation between the Green functions

$$G^{\lambda}_{mn}(x_b,\tilde{f}_b,t_b;x_a,\tilde{f}_a,t_a)=\int_{\mathcal{G}} G_{\tilde{\mathcal{P}}}(p_b\theta,v_b\theta,t_b;p_a,t_a)D^{\lambda}_{nm}(\theta)d\mu(\theta),$$

$$(x,\tilde{f})=\pi'(p,v).$$

G^{λ}_{mn} is written symbolically as

$$G^{\lambda}_{mn}(\pi'(p_b),\pi'(v_b),t_b;\pi'(p_a),\pi'(v_a),t_a)=$$

$$\tilde{\mathrm{E}}_{\substack{\xi(t_a)=\pi'(p_a,v_a)\\ \xi(t_b)=\pi'(p_b,v_b)}}\Big[(\overleftarrow{\exp})^{\lambda}_{mn}(\xi(t),t_b,t_a)\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(\xi_1(u),\xi_2(u))du\}\Big]$$

$$=\int\limits_{\substack{\xi(t_a)=\pi'(p_a,v_a)\\ \xi(t_b)=\pi'(p_b,v_b)}} d\mu^{\xi}\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u),\tilde{f}(u))du\}$$

$$\times\overleftarrow{\exp}\int_{t_a}^{t_b}\Big\{\mu^2\kappa\Big[\frac{1}{2}d^{\alpha\nu}(x(u),\tilde{f}(u))(J_{\alpha})^{\lambda}_{mr}(J_{\nu})^{\lambda}_{rn}$$

$$-\frac{1}{2}\frac{1}{\sqrt{d\,H}}\frac{\partial}{\partial x^k}\left(\sqrt{d\,H}h^{km}\underset{(\gamma)}{\mathcal{A}^{\nu}_{m}}(x(u))\right)(J_{\nu})^{\lambda}_{mn}-\ldots\Big]\Big\}$$

The semigroup with this kernel acts in the space of th equivariant functions on $\tilde{\mathcal{P}}$:

$$\tilde{\psi}_n(pg, vg) = D^{\lambda}_{mn}(g)\tilde{\psi}_n(p, v).$$

The isomorphism of these functions with the functions $\psi_n \in \Gamma(\tilde{\mathcal{M}}, V^*)$ is given by

$$\tilde{\psi}_n(F(Q^*(x), e), \bar{D}^b_c(e)\tilde{f}^c) = \psi_n(x, \tilde{f}).$$

Reduction to zero-momentum level: ($\lambda = 0$ - case)

The result of the Girsanov transformation:

$$\frac{d\mu^{\xi}}{d\mu^{\tilde{\xi}}}(\tilde{\xi}(t)) = \left(\frac{exp(\sigma(x(t_b),\tilde{f}(t_b))}{exp(\sigma(x(t_a),\tilde{f}(t_a))}\right)^{1/4} \times \exp\left\{-\frac{1}{8}\mu^2\kappa\int_{t_a}^{t_b}(\triangle_{\tilde{\mathcal{M}}}\sigma+\frac{1}{4}<\partial\sigma,\partial\sigma>_{\tilde{\mathcal{M}}})du\right\}$$

$\sigma_i = \frac{\partial}{\partial x^i}(\ln d)$, $\sigma_a = \frac{\partial}{\partial \tilde{f}^a}(\ln d)$ and $<\partial\sigma,\partial\sigma>_{\tilde{\mathcal{M}}}$ is

$$\left[h^{ij}\sigma_i\sigma_j + 2h^{kj}\underset{(\gamma)}{\mathcal{A}^{\mu}_k}K^a_{\mu}\sigma_a\sigma_j + \left((\gamma^{\alpha\beta}+h^{kl}\underset{(\gamma)}{\mathcal{A}^{\alpha}_k}\underset{(\gamma)}{\mathcal{A}^{\beta}_l})K^a_{\alpha}K^b_{\beta}+G^{ab}\right)\sigma_a\sigma_b\right].$$

The following integral relation:

$$d_b^{-1/4}d_a^{-1/4}G_{\tilde{M}}(x_b,\tilde{f}_b,t_b;x_a,\tilde{f}_a,t_a) = \int_{\mathcal{G}} G_{\tilde{\mathcal{P}}}(p_b\theta,v_b\theta,t_b;p_a,v_a,t_a)d\mu(\theta),$$

$d_b = d(x_b,\tilde{f}_b)$, $d_a = d(x_a,\tilde{f}_a)$.

$$
\begin{aligned}
&G_{\tilde{\mathcal{M}}}(x_b, \tilde{f}_b, t_b; x_a, \tilde{f}_a, t_a) \\
&\qquad = \int_{\substack{\tilde{\xi}(t_a)=(x_a,\tilde{f}_a) \\ \tilde{\xi}(t_b)=(x_b,\tilde{f}_b)}} d\mu^{\tilde{\xi}} \exp\left\{\frac{1}{\mu^2 \kappa m}\int_{t_a}^{t_b} V(\tilde{\xi}_1(u), \tilde{\xi}_2(u))du\right\} \\
&\qquad \times \exp\Big\{-\frac{1}{8}\mu^2\kappa \int_{t_a}^{t_b} (\triangle_{\tilde{\mathcal{M}}}\sigma + \frac{1}{4} < \partial\sigma, \partial\sigma >_{\tilde{\mathcal{M}}})du\Big\}, \\
&\qquad (x, \tilde{f}) = \pi'(p, v), \;\; \sigma = \sigma(\tilde{\xi}_1(u), \tilde{\xi}_2(u)).
\end{aligned}
$$

$G_{\tilde{\mathcal{M}}}$ satisfies the forward equation by the variables $(x_b, \tilde{f}_b, t_b)$. The operator of the forward Kolmogorov equation is

$$
\hat{H}_\kappa = \frac{\hbar\kappa}{2m}\triangle_{\tilde{\mathcal{M}}} - \frac{\hbar\kappa}{8m}\Big[\triangle_{\tilde{\mathcal{M}}}\sigma + \frac{1}{4} < \partial\sigma, \partial\sigma >_{\tilde{\mathcal{M}}}\Big] + \frac{1}{\hbar\kappa}V.
$$

At $\kappa = i \Rightarrow$ the Schrödinger equation with $\hat{H} = -\frac{\hbar}{\kappa}\hat{H}_\kappa\big|_{\kappa=i}$.

The geometrical representation of the reduction Jacobian

$$
\mathrm{R}_{\tilde{\mathcal{P}}} = \mathrm{R}_{\tilde{\mathcal{M}}} + \mathrm{R}_{\mathcal{G}} + \frac{1}{4}\tilde{h}^{A'B'}\tilde{h}^{C'D'}d_{\mu\nu}\mathcal{F}^{\mu}_{A'C'}\mathcal{F}^{\nu}_{B'D'}
$$
$$
+\frac{1}{4}\tilde{h}^{A'B'}d^{\mu\sigma}d^{\nu\kappa}(\mathcal{D}_{A'}d_{\mu\nu})(\mathcal{D}_{B'}d_{\sigma\kappa}) + \triangle_{\tilde{\mathcal{M}}}\ln d + \frac{1}{4}G_{\tilde{\mathcal{M}}}(\partial\ln d, \partial\ln d).
$$

$(A' = (i, a),)$ $\mathcal{D}_{A'}d_{\mu\nu} = \partial_{A'}d_{\mu\nu} - c^{\kappa}_{\sigma\mu}\mathcal{A}^{\sigma}_{A'}d_{\kappa\nu} - c^{\kappa}_{\sigma\nu}\mathcal{A}^{\sigma}_{A'}d_{\mu\kappa}$

The second fundamental form of the orbit

$$
j_{\alpha\beta} = -\frac{1}{2}\Big[(\tilde{h}^{kn}\mathcal{D}_k d_{\alpha\beta} + \tilde{h}^{bn}\mathcal{D}_b d_{\alpha\beta})\,\hat{H}_n + (\tilde{h}^{kb}\mathcal{D}_k d_{\alpha\beta} + \tilde{h}^{ab}\,D_a d_{\alpha\beta})\,\hat{H}_b].
$$

Terms with covariant derivatives—the trace of the square $j_{\alpha\beta}|_{\tilde{\mathcal{M}}}$

$$
G_{\tilde{\mathcal{M}}}(j_{\alpha\beta}, j_{\mu\nu})d^{\alpha\beta}d^{\mu\nu} =
$$
$$
\frac{1}{4}d^{\alpha\beta}d^{\mu\nu}\Big[(\mathcal{D}_k d_{\alpha\beta})(\mathcal{D}_l d_{\mu\nu})\tilde{h}^{kl} + (\mathcal{D}_b d_{\alpha\beta})(\mathcal{D}_l d_{\mu\nu})\tilde{h}^{bl}
$$
$$
+(\mathcal{D}_k d_{\alpha\beta})(\mathcal{D}_d d_{\mu\nu})\tilde{h}^{dk} + (\mathcal{D}_a d_{\alpha\beta})(\mathcal{D}_d d_{\mu\nu})\tilde{h}^{ad}\Big],
$$

The Hamilton operator of the Schrödinger equation on the reduced manifold $\tilde{\mathcal{M}}$:

$$\hat{H}_{\tilde{\mathcal{M}}} = -\frac{\hbar^2}{2m}\triangle_{\tilde{\mathcal{M}}} + \frac{\hbar^2}{8m}\Big[\mathrm{R}_{\tilde{\mathcal{P}}} - \mathrm{R}_{\tilde{\mathcal{M}}} - \mathrm{R}_{\mathcal{G}} - \frac{1}{4}d_{\mu\nu}\mathcal{F}^{\mu}_{A'B'}\mathcal{F}^{\nu A'B'} - ||j||^2\Big] + \tilde{V}.$$