# Quark Density in Lattice $QC_2D$ at Imaginary and Real Chemical Potential

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Our goal:

To find the best parametrization of the quark density
for its analytical continuation
from imaginary to real quark chemical potential

# Outline

1. Simulation settings
2. Analytical continuation of the quark density
3. Cluster Expansion Model (CEM) vs Ratiomal Fraction Model (RFM)
4. CEM and the fugacity expansion
5. Conclusions

# Parameters of simulation

- Tree-level inproved Symanzik gauge action
- Staggered fermions with a diquark source ( diquark coupling $\lambda = 0.00125$)

Sommer parameter $r_0 = 0.468$fm
Lattice spacing $a \approx 0.062$ fm
Lattice size $L \approx 1.74$ fm
$am_q = 0.0125$; $m_\pi \approx 800$ MeV

$N_c = 2, \quad N_f = 2$
$N_s^3 \times N_t$ lattices: $N_s = 28$;
$N_t = 14, \quad 12$
$T = 227, \; 265$ MeV

$$\theta = \frac{\mu_q}{T} = \frac{\mu'_q + \imath\mu''_q}{T} = \theta_R + \imath\theta_I$$

$$0 \leq \theta_I \leq \frac{\pi}{N_c}, \qquad 0 < \mu'_q < 600 \text{ MeV}$$

$$S_G = \beta \left( 1.824 \sum_{\square} \left( 1 - \frac{1}{2} \mathrm{Tr} \square \right) - 0.165 \sum_{\square\square} \left( 1 - \frac{1}{2} \mathrm{Tr} \square\square \right) \right) \quad (1)$$

$$S_F = \sum_{x,y} \bar{\psi}_x D(\mu_q)_{x,y} \psi_y + \frac{\lambda}{2} \sum_x \left( \psi_x^T \tau_2 \psi_x + \bar{\psi}_x \tau_2 \bar{\psi}_x^T \right) \quad (2)$$

where $\bar{\psi}$, $\psi$ are staggered fermion fields,

$$D(\mu_q)_{xy} = ma\delta_{xy} + \frac{1}{2} \sum_{\nu=1}^{4} \eta_\nu(x) \Big[ U_{x,\nu} \delta_{x+h_\nu,y} e^{\mu_q a \delta_{\nu,4}} - U^\dagger_{x-h_\nu,\nu} \delta_{x-h_\nu,y} e^{-\mu_q a \delta_{\nu,4}} \Big], \quad (3)$$

$\eta_1(x) = 1$, $\eta_\nu(x) = (-1)^{x_1+\ldots+x_{\nu-1}}$, $\nu = 2, 3, 4$.

$$\text{We use } B = \frac{n_q V}{N_c} \text{ instead of } n_q$$

$B$ is the baryon number in the lattice volume,

$$\begin{aligned} B(\theta) &= \frac{1}{N_c} \frac{\partial \ln Z_{GC}(\theta)}{\partial \theta} \\ &= \frac{N_f}{4 N_c Z_{GC}} \int \mathcal{D}U e^{-S_G} (\det M)^{N_f/8} \mathrm{tr} \left[ M^{-1} \frac{\partial M}{\partial \theta} \right] , \end{aligned}$$

where $M = D^\dagger(\mu_q) D(\mu_q) + \lambda^2$ and

$$Z_{GC}(\theta) = \int \mathcal{D}U e^{-S_G} (\det M)^{N_f/8} \tag{4}$$

is the Grand Canonical (GC) partition function.

# Properties of the grand canonical partition function

$$Z_{GC}(\theta, T, V) = \sum_n \langle n| \exp\left(\frac{-\hat{H} + \mu\hat{Q}}{T}\right)|n\rangle \tag{5}$$

meets the fugacity expansion, that is the Laurent series in $\xi = e^{\theta}$:

$$Z_{GC}(\theta, T, V) = \sum_{k=-\infty}^{\infty} Z_C(kN_c, T, V) e^{kN_c\theta}, \tag{6}$$

it involves powers of $\xi^{N_c}$ owing to the Roberge-Weiss symmetry

$$Z_{GC}(\theta_I, T, V) = Z_{GC}(\theta_I + 2\pi/N_c, T, V) \;, \tag{7}$$

$C$−parity $\implies$ $Z_{GC}(\theta_I, T, V) = Z_{GC}(-\theta_I, T, V)$

- Problem:
  The quark density $n_q(\theta)$ cannot be determined
  in lattice QCD at $\theta = \theta_R$ because of the sign problem.
- Solution:
  Find it at $\theta = \imath\theta_I$ and then employ analytical continuation in $\theta$

- Problem in this way:
  Analytical continuation in $\theta$ depends on parametrization of $n_q(\theta)$
- Proposed solution:
  Test different parametrizations in the case of $QC_2D$,
  where $n_q(\theta)$ can be simulated at both $\theta = \theta_R$ and $\theta = \imath\theta_I$

# Naive analytic continuation

Assuming that

$$B(\theta)\Big|_{\theta_R=0} = \imath \sum_{n=1}^{\infty} a_n \sin\left(nN_c\theta_I\right), \tag{8}$$

we arrive at

$$B(\theta)\Big|_{\theta_I=0} = \sum_{n=1}^{\infty} a_n \sinh\left(nN_c\theta_R\right) \tag{9}$$

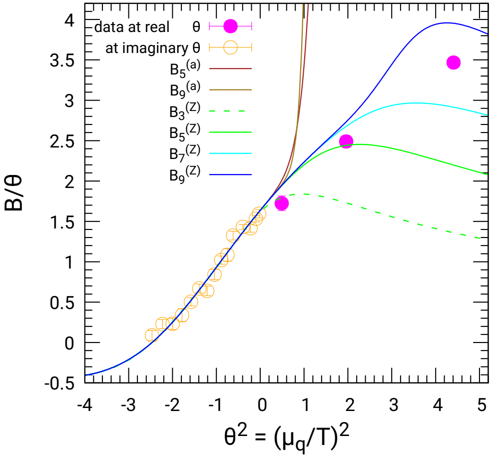
Limitatons:

- $a_n$ are extracted from a fit over the segment $0 \le \theta_I \le \dfrac{\pi}{N_c}$

  $\Longrightarrow$ only a few of $a_n$ can be determined.

- Series (9) converges only if $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = r$ exists

  and $|\theta_R| < \dfrac{-\ln r}{N_c}$




$T = 227$ MeV

$$B_J^{(a)} = \frac{2\sum_{n=1}^{J} nZ_n \sin(nN_c\theta_I)}{1 + 2\sum_{n=1}^{J} Z_n \cos(nN_c\theta_I)}$$

# Rational Fraction Model (RFM)

[G. A. Almasi, B. Friman, K. Morita, P. M. Lo, and K. Redlich 2019]

$$B(\theta_I)\Big|_{\theta_R=0} = \sum_{k=1}^{\infty} a_k^{\mathrm{RFM}} \sin\left(kN_c\theta_I\right) \tag{10}$$

$$a_n^{\mathrm{RFM}} = (-1)^{n+1}\, d\, \frac{1+\dfrac{\pi^2(N_c^2-1)}{6}n^2}{n^3(1+n\kappa)}\,. \tag{11}$$

$a_n^{\mathrm{RFM}} \sim \dfrac{(-1)^k}{k^2}$ as $k\to\infty \implies$ nonanalytic behavior:

$$B(\theta) \sim \left(\theta_I - \frac{\pi}{N_c}\right) \ln\left(\frac{\pi}{N_c} - \theta_I\right) \qquad \text{as} \qquad \theta_I \to \frac{\pi}{N_c} \tag{12}$$

$$B_{RFM}(\theta) = d \left\{ \left( \frac{\pi^2(N_c^2-1)}{6} + \kappa^2 \right) \left[ \frac{\theta N_c}{2} - \right. \right. \tag{13}$$

$$\left. - \left( \beta\left(\frac{1}{\kappa}\right) - \frac{\kappa}{2} \right) \sinh\left(\frac{\theta N_c}{\kappa}\right) + \frac{1}{2}\int_0^{\theta N_c} dt\ \tanh\frac{t}{2} \sinh\frac{\theta N_c - t}{\kappa} \right]$$

$$\left. + \frac{\pi^2}{12}\left(\theta N_c + \frac{(\theta N_c)^3}{\pi^2}\right) - \kappa \int_0^{\theta N_c} \ln\left(2\cosh\frac{t}{2}\right) dt \right\}$$

where

$$\beta(z) = \frac{1}{2}\left(\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right), \qquad \psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}$$

# Cluster Expansion Model (CEM)

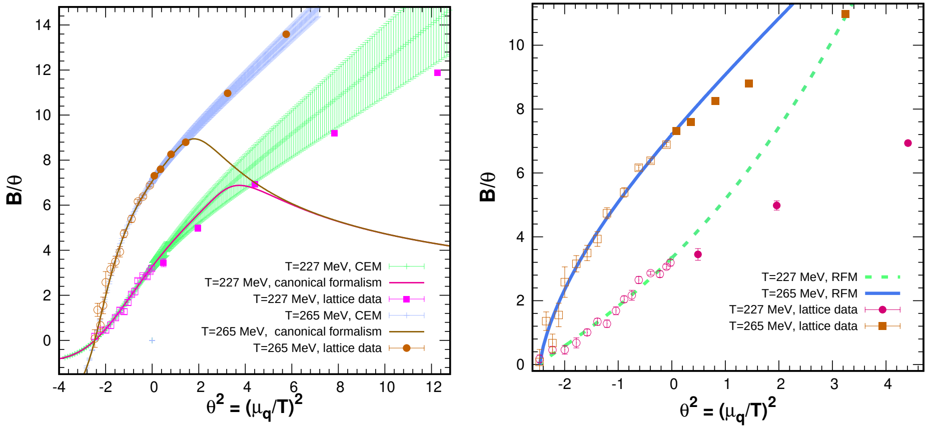
[V. Vovchenko, J. Steinheimer, O. Philipsen and H. Stoecker 2018]

$$B(\theta_I)\Big|_{\theta_R=0} = \sum_{k=1}^{\infty} a_k^{\text{CEM}} \sin(kN_c\theta_I) \tag{14}$$

$$b_k = (-1)^{k+1} \frac{b\,q^{k-1}}{k} \left[1 + \frac{6}{\pi^2(N_c^2-1)k^2}\right] \tag{15}$$

$$B = \frac{b}{2q}\left\{\ln\frac{1+q\exp(\theta N_c)}{1+q\exp(-\theta N_c)} + \right.$$
$$\left. + \frac{6}{\pi^2(N_c^2-1)}\left[\mathrm{Li}_3\left(-qe^{-\theta N_c}\right) - \mathrm{Li}_3\left(-qe^{\theta N_c}\right)\right]\right\}\ . \tag{16}$$

# Comparison of the CEM and RFM with lattice data



Fugacity expansion

$$\frac{Z_{GC}(\theta, T, V)}{Z_C(0, T, V)} = 1 + \sum_{n=1}^{\infty} Z_n \left( e^{nN_c\theta} + e^{-nN_c\theta} \right) \tag{17}$$

provides a natural parametrization of $B(\theta)$,

$$B(\theta) = \frac{-1}{N_c} \frac{\partial (T \ln Z)}{\partial \mu_q} = \frac{2 \sum_{n=1}^{\infty} n Z_n \sinh(nN_c\theta)}{1 + 2\sum_{n=1}^{\infty} Z_n \cosh(nN_c\theta)} \tag{18}$$

$$B(\theta_I)\Big|_{\theta_R=0} = \imath \sum_{n=1}^{\infty} a_n \sin\left(nN_c\theta_I\right) \tag{19}$$

$$\sum_{n=1}^{\infty} a_n \sin\left(nN_c\theta_I\right) = \frac{2 \sum_{n=1}^{\infty} n Z_n \sin(nN_c\theta_I)}{1 + 2\sum_{n=1}^{\infty} Z_n \cos(nN_c\theta_I)} \tag{20}$$

Problem: Given $a_n$, find $Z_n$

$$\text{Trigonometric identities} \quad \Longrightarrow \quad a_i = \sum_{j=1}^{\infty} W_{ij} Z_j \;, \tag{21}$$

$$W_{jk} = 2j\delta_{jk} - a_{j+k} + a_{|j-k|} \cdot \text{sign}(k-j) \qquad\qquad [\text{sign}(0) = 0] \;. \tag{22}$$

$$\mathbf{Z} = \mathbf{W}^{-1}\mathbf{a} \,. \tag{23}$$

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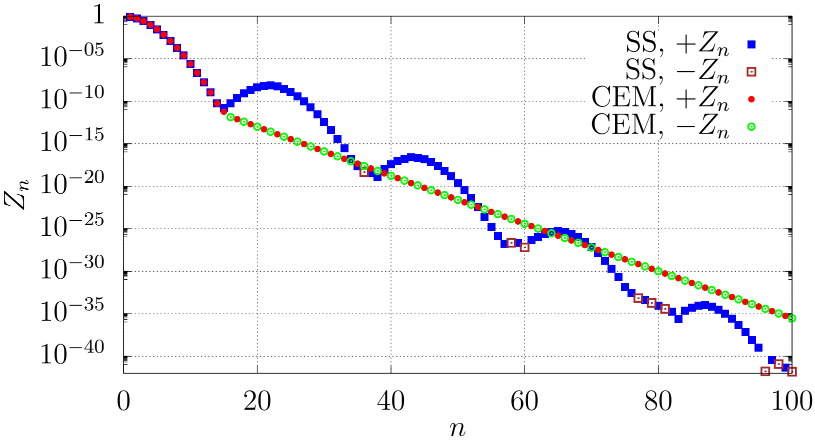
$$Z_{GC}(\theta_I) = \exp\left( N_c \sum_{n=1}^{N} \frac{a_n}{2n} \Big( \cos(nN_c\theta_I) - 1 \Big) \right) \tag{24}$$

The inverse of the fugacity expansion has the form

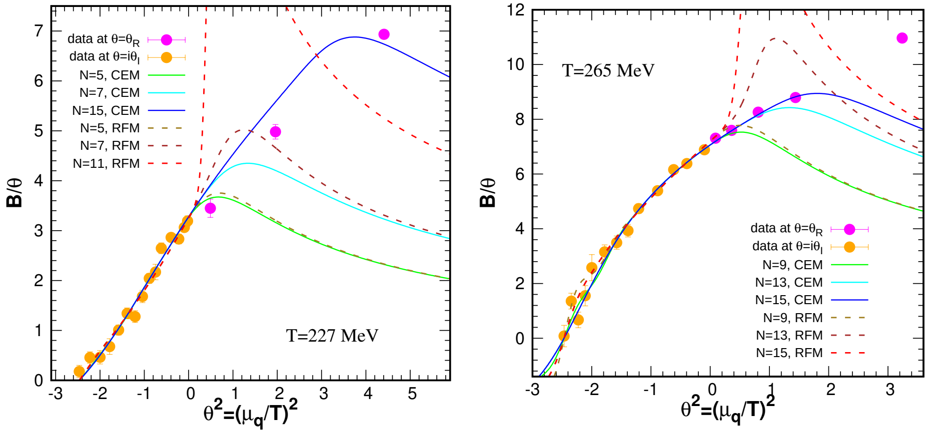
$$Z_C\,(n,T,V) = \int_0^{2\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I,T,V) \;, \tag{25}$$




SS - $Z_n$ from the truncared Fourier series;

CEM - $Z_n$ found using analytic formula; Empty symbols: $Z_n < 0$

# Comparison of the fugacity expansions using CEM and RFM



# Conclusions

We have studied the analytical continuation of the quark density in $QC_2D$ at $T < T_{RW}$ using various parametrizations. It was found

| theoretical framework of parametrization | Agreement of the respective analytical continuation with lattice data at real $\mu_q$ |
|---|---|
| truncated Fourier series | bad |
| CEM | excellent |
| RFM | poor |
| the grand canonical approach with the CEM | good at $\lvert\mu_q\rvert < 320 \div 390$ MeV |

Problem of negative canonical partition functions $Z_C(n, T, V)$ calls for further work