Weights and characters

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Lie algebras and their representations

Lie algebra

A vector space $\mathfrak{g}$ together with a bilinear operation $[\,\cdot\,,\,\cdot\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket and having the properties

$$[x, y] = -[y, x],$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

for all $x, y, z \in \mathfrak{g}$ is called a Lie algebra.

- The simplest nontrivial example is the Lie algebra $\mathfrak{sl}_2$. It is the set of linear combinations of three elements H, E, F. The Lie bracket is defined by the equations

$$[H, E] = 2E, \qquad [H, F] = 2F,$$
$$[E, F] = H.$$

- For this talk we assume that a vector space is a vector space over the field $\mathbb{C}$ of complex numbers, and an algebra is a complex associative algebra with unit.

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$$[x,\,y] = -[y,\,x],$$
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- For this talk we assume that a vector space is a vector space over the field $\mathbb{C}$ of complex numbers, and an algebra is a complex associative algebra with unit.

Representation

A representation π of a Lie algebra $\mathfrak{g}$ on a linear space V is defined as a linear mapping from $\mathfrak{g}$ to the algebra $\mathrm{End}(V)$ of linear operators on V, satisfying the equation

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x).$$

Module

A linear space V is said to be a $\mathfrak{g}$-module if a bilinear mapping $\cdot : \mathfrak{g} \times V \to V$, satisfying the equation

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v),$$

is given.

- Given a representation π of $\mathfrak{g}$ on a linear space V, we define on V the structure of a $\mathfrak{g}$-module by the equation

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and vise versa.

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  and vise versa.

- Let V be an $\mathfrak{sl}_2$-module, and v_λ a vector, satisfying the equation

$$H \cdot v_\lambda = \lambda v_\lambda.$$

It is easy to demonstrate that

$$H \cdot (E \cdot v_\lambda) = (\lambda + 2) v_\lambda, \qquad H \cdot (F \cdot v_\lambda) = (\lambda - 2) v_\lambda.$$

- Assume additionally that

$$E \cdot v_\lambda = 0,$$

and denote

$$v_0 = v_\lambda, \qquad v_n = F \cdot v_{n-1}, \qquad n = 1, 2, \dots,$$

Assume that all v_n are linearly independent and denote the space generated by them as $\widetilde{V}^\lambda$.

- We obtain

$$H \cdot v_n = (\lambda - 2n) v_n,$$

$$F \cdot v_n = v_{n+1}, \qquad E \cdot v_n = n(\lambda - n + 1) v_{n-1}.$$

Here $v_{-1} = 0$.

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is infinite dimensional. However, let $\lambda = m$ be a nonnegative integer. In this case

$$E \cdot v_{m+1} = (m+1)(m - (m+1) + 1)v_m = 0.$$

It is clear that

$$H \cdot v_{m+1} = (-m-2)v_{m+1}.$$

- Thus, the space generated by the vectors v_n with $n = m+1, \ldots$ is an $\mathfrak{sl}_2$-submodule of $\widetilde{V}^m$ isomorphic to $\widetilde{V}^{-m-2}$.
- The quotient module

$$V^m = \widetilde{V}^m / \widetilde{V}^{-m-2}$$

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Definition

A square matrix $(a_{ij})_{i,j=1,\dots,r}$ satisfying the conditions

- $a_{ii} = 2$ for $i = 1, \dots r$
- a_{ij} are nonpositive integers for $i \neq j$
- $a_{ij} = 0$ implies $a_{ji} = 0$
- all principal minors of (a_{ij}) are positive

is called a Cartan matrix.

- For example, for $r = 1$ we have

$$\underset{\mathfrak{sl}_2}{(a_{ij}) = (2),}$$

and for $r = 2$ there are three nontrivial possibilities,

$$\underset{\mathfrak{sl}_3}{(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}, \quad \underset{\mathfrak{so}_5}{(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}}, \quad \underset{\mathfrak{sp}_2}{(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}}.$$

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- Consider the Lie algebra generated by $3r$ generators H_i, E_i and F_i satisfying the relations

$$[H_i, H_j] = 0,$$
$$[H_i, E_j] = a_{ij} E_j, \qquad [H_i, F_j] = -a_{ij} F_j,$$
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Adjoint representaion

A mapping $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined as

$$\mathrm{ad}(X)Y = [X, Y]$$

is a representation of $\mathfrak{g}$ on $\mathfrak{g}$ called the adjoint representation.

- The Serre relations look as

$$\mathrm{ad}(E_i)^{-a_{ij}+1}E_j = 0, \qquad \mathrm{ad}(F_i)^{-a_{ij}+1}F_j = 0$$

  for all $i \neq j$.
- There is no Serre relations for the Lie algebra $\mathfrak{sl}_2$. The Serre relations for the Lie algebra $\mathfrak{sl}_3$ are

$$[E_1, [E_1, E_2]] = 0, \qquad [E_2, [E_2, E_1]] = 0,$$
$$[F_1, [F_1, F_2]] = 0, \qquad [F_2, [F_2, F_1]] = 0.$$

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Theorem

For any Cartan matrix (a_{ij}) the relations

$$[H_i, H_j] = 0,$$
$$[H_i, E_j] = a_{ij}E_j, \qquad [H_i, F_j] = -a_{ij}F_j,$$
$$[E_i, F_j] = \delta_{ij}H_i,$$
$$\mathrm{ad}(E_i)^{-a_{ij}+1}E_j = 0, \qquad \mathrm{ad}(F_i)^{-a_{ij}+1}F_j = 0, \qquad i \neq j.$$

define a simple finite dimensional Lie algebra.

- From now on $\mathfrak{g}$ is a simple finite dimensional Lie algebra of rank r.

Weight vectors

An element v of a $\mathfrak{g}$-module V is called a weight vector of weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ if

$$H_i \cdot v = \lambda_i v, \qquad i = 1, \ldots, r.$$

We denote

$$V_\lambda = \{v \in V \mid H_i \cdot v = \lambda_i v, \quad i = 1, \ldots, r\}.$$

Whenever $V_\lambda \neq \{0\}$, we call V_λ the weight space of weight λ.

Cartan subalgebra

The linear span

$$\mathfrak{h} = \bigoplus_{i=1}^{r} \mathbb{C} H_i$$

is an abelian subalgebra of $\mathfrak{g}$ called the Cartan subalgebra.

- We identify $\lambda = (\lambda_1, \ldots, \lambda_r)$ with an element of the dual space $\mathfrak{h}^*$ such that

$$\langle \lambda, H_i \rangle = \lambda_i, \qquad i = 1, \ldots, r.$$

- With such definition, one can rewrite the definition of a weight space as

$$V_\lambda = \{ v \in V \mid H \cdot v = \langle \lambda, H \rangle v, \quad H \in \mathfrak{h} \}.$$

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Weight module

A $\mathfrak{g}$-module such that

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

is said to be a weight module.

- Any finite dimensional $\mathfrak{g}$-module is a weight module. An infinite dimensional $\mathfrak{sl}_2$-module $\widetilde{V}^m$ discussed above is a weight module.

Higest weight module

A $\mathfrak{g}$-module V is said to be a highest weight module of highest weight λ if it is generated by a vector v such that

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Definition

Let V be a highest weight $\mathfrak{g}$-module such that all weight spaces V_λ are finite dimensional. The character of V is defined as the formal sum

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda \, \mathrm{e}^\lambda,$$

where e^λ, $\lambda \in \mathfrak{h}$ a formal objects satisfying the equation

$$\mathrm{e}^{\lambda_1}\mathrm{e}^{\lambda_2} = \mathrm{e}^{\lambda_1+\lambda_2}.$$

- Two highest weight $\mathfrak{g}$-modules V and W are isomorphic if and only if $\mathrm{ch}(V) = \mathrm{ch}(W)$.

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Direct sum of $\mathfrak{g}$-modules

Let V and W be $\mathfrak{g}$-modules. The $\mathfrak{g}$-module operation for $V \oplus W$ is defined by the equation

$$x \cdot (v, w) = (x \cdot v, x \cdot w).$$

- For any two highest weight $\mathfrak{g}$-modules V and W one has

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W).$$

Tensor product of $\mathfrak{g}$-modules

Let V and W be $\mathfrak{g}$-modules. The $\mathfrak{g}$-module operation for $V \times W$ is defined by the equation

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

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- For the infinite dimensional $\mathfrak{sl}_2$-module $\widetilde{V}^\lambda$ one has

$$\mathrm{ch}(\widetilde{V}^\lambda) = \mathrm{e}^\lambda + \mathrm{e}^{\lambda-2} + \mathrm{e}^{\lambda-4} + \cdots = \mathrm{e}^\lambda(1 + \mathrm{e}^{-2} + \mathrm{e}^{-4} + \cdots) = \frac{\mathrm{e}^\lambda}{1-\mathrm{e}^{-2}}.$$

- For the finite dimensional $\mathfrak{sl}_2$-module V^m one has

$$\mathrm{ch}(V^m) = \mathrm{e}^m + \mathrm{e}^{m-2} + \cdots \mathrm{e}^{-m} = \mathrm{e}^m(1 + \mathrm{e}^{-2} + \cdots + \mathrm{e}^{-2m})$$

$$= \mathrm{e}^m \frac{1-\mathrm{e}^{-2m-2}}{1-\mathrm{e}^{-2}} = \frac{\mathrm{e}^{m+1} - \mathrm{e}^{-m-1}}{\mathrm{e}-\mathrm{e}^{-1}}$$

- For the infinite dimensional $\mathfrak{sl}_2$-module $\widetilde{V}^\lambda$ one has

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- For the tensor product $V^m \otimes V^n$ we obtain

$$\begin{aligned}\mathrm{ch}(V^m \otimes V^n) &= \frac{e^{m+1} - e^{-m-1}}{e - e^{-1}} \cdot \frac{e^{n+1} - e^{-n-1}}{e - e^{-1}} \\ &= \frac{(e^{m+1} - e^{-m-1})(e^{n+1} - e^{-n-1})}{(e - e^{-1})^2} = \frac{e^{m+n+1} - e^{-m-n-1}}{e - e^{-1}} \\ &\quad + \frac{e^{m+n-1} - e^{-m-n+1}}{e - e^{-1}} + \cdots + \frac{e^{m-n+1} - e^{-m+n-1}}{e - e^{-1}}.\end{aligned}$$

- It follows that

$$\mathrm{ch}(V^m \otimes V^n) = \mathrm{ch}(V^{m+n}) + \mathrm{ch}(V^{m+n-2}) + \cdots + \mathrm{ch}(V^{m-n}).$$

- Thus, we have

$$V^m \otimes V^n \simeq V^{m+n} \oplus V^{m+n-2} \oplus \cdots \oplus V^{m-n}.$$

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Quantum algebras

Problem

One cannot multiply elements of a Lie algebra.

Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra, and $X_1, X_2, \ldots, X_N$ its basis so that

$$[X_i, X_j] = \sum_{k=1}^{N} c_{ijk} X_k.$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is an associative algebra with unit generated by generators $x_1, x_2, \ldots, x_N$ subject to the relations

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and no other relations.

- The elements $X_1, X_2, \ldots, X_N$ form a basis of $\mathfrak{g}$. The elements $x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}$ form a basis of $\mathrm{U}(\mathfrak{g})$.

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- A simple finite dimensional Lie algebra $\mathfrak{g}$ via generators and relations:

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\begin{gathered}
[H_i, H_j] = 0, \\
[H_i, E_j] = a_{ij}E_j, \qquad [H_i, F_j] = -a_{ij}F_j, \\
[E_i, F_j] = \delta_{ij}H_i, \\
\mathrm{ad}(E_i)^{-a_{ij}+1}E_j = 0, \qquad \mathrm{ad}(F_i)^{-a_{ij}+1}F_j = 0, \qquad i \neq j.
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- Given a Lie algebra $\mathfrak{g}$. The quantum algebra is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. Denote the deformation parameter by $\hbar$, and assume that $q = \mathrm{e}^{\hbar} \neq \pm 1$.
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Quantum algebras

- Any Cartan matrix is symmetrizable. This means that there exist unique co-prime positive integers d_i such that the matrix $(d_i a_{ij})$ is symmetric.

Quantum algebra

The quantum algebra $\mathrm{U}_q(\mathfrak{g})$ is an associative algebra with unit generated by q^x, $x \in \mathfrak{h}$, e_i, f_i, $i = 1, \ldots, r$, subject to the relations

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$$\Delta\colon \mathrm{U}_q(\mathfrak{g}) \to \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g}).$$

We have

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$$m(b \otimes a) = ba = ab = m(a \otimes b).$$

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- The terminology used for $U_q(\mathfrak{g})$-modules is almost the same as for $\mathfrak{g}$-modules except the definition of a weight vector.

Weight vector

An element v of a $U_q(\mathfrak{g})$-module V is called a weight vector of weight $\lambda \in \mathfrak{h}^*$ if

$$q^x v = q^{\langle \lambda, x \rangle} v.$$

We denote

$$V_\lambda = \{v \in V \mid q^x v = q^{\langle \lambda, x \rangle} v\}.$$

Whenever $V_\lambda \neq \{0\}$, we call V_λ the weight space of weight λ.

- The formulas

$$q^{\nu h} \cdot v_n = q^{\nu(m-2n)} v_n,$$
$$f \cdot v_n = v_{n+1}, \qquad e \cdot v_n = [n]_q \, [m-n+1]_q v_{n-1},$$

where $v_{m+1} = 0$ and $v_{-1} = 0$, describe the structure of an $(m+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module. We denote it again as V^m.

- Let V be a highest weight $U_q(\mathcal{L}(\mathfrak{g}))$-module such that all weight spaces V_λ are finite dimensional. The character of V is defined as the formal sum

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda \, \mathrm{e}^\lambda,$$

where e^λ, $\lambda \in \mathfrak{g}^*$, are formal objects satisfying the equation

$$\mathrm{e}^{\lambda_1} \mathrm{e}^{\lambda_2} = \mathrm{e}^{\lambda_1 + \lambda_2}.$$

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Application to integrable systems

- We study integrable systems based on quantum algebras $U_q(\mathcal{L}(\mathfrak{g}))$, where $\mathcal{L}(\mathfrak{g})$ is the loop algebra of a Lie algebra $\mathfrak{g}$.

- Let φ be a representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$ on a space V and ψ a representation on a space W. The corresponding integrability object is defined as

$$X_{\varphi|\psi} = (\varphi \otimes \psi)(\mathcal{R}) \in \mathrm{End}(V) \otimes \mathrm{End}(W) \simeq \mathrm{End}(V \otimes W).$$

In this way, one constructs R-operators, L-operators and monodromy operators.

- Starting from an integrability object $X_{\varphi|\psi}$ one defines an integrability object

$$Y_{\varphi|\psi} = (\mathrm{tr}_V \otimes \mathrm{id}_W) X_{\varphi|\psi} \in \mathrm{End}(W).$$

In this way, one constructs transfer operators and Q-operators.

- An important property of the integrability objects $Y_{\varphi|\psi}$ is that if two representations φ_1 and φ_2 are isomorphic then

$$Y_{\varphi_1|\psi} = Y_{\varphi_2|\psi}.$$

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$$Y_{\varphi|\psi} = (\mathrm{tr}_V \otimes \mathrm{id}_W) X_{\varphi|\psi} \in \mathrm{End}(W).$$

  In this way, one constructs transfer operators and Q-operators.

- An important property of the integrability objects $Y_{\varphi|\psi}$ is that if two representations φ_1 and φ_2 are isomorphic then

$$Y_{\varphi_1|\psi} = Y_{\varphi_2|\psi}.$$

- Let φ be a representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$ on a space V and ψ a representation on a space W. The corresponding integrability object is defined as

$$X_{\varphi|\psi} = (\varphi \otimes \psi)(\mathcal{R}) \in \mathrm{End}(V) \otimes \mathrm{End}(W) \simeq \mathrm{End}(V \otimes W).$$

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- Recall that $\mathcal{R} \in \mathrm{U}_q(\mathcal{L}(\mathfrak{g})) \otimes \mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$. By agreement, the first factor of that tensor product is associated with the type of the integrable object and the second one with the concrete integrable model. Universal integrability objects describe the entire class of integrable models associated with a given quantum algebra. Here we have universal integrability objects defined as

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and as

$$\mathcal{Y}_\varphi = ((\mathrm{tr}_V \circ \varphi) \otimes \mathrm{id}_{\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))})\mathcal{X}_\varphi \in \mathrm{U}_q(\mathcal{L}(\mathfrak{g})).$$

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Tensor product of mappings

Let φ_1 and φ_2 be linear mappings from vector spaces V_1 and V_2 to vector spaces W_1 and W_2 respectively. We define the tensor product of φ_1 and φ_2 as a mapping from $V_1 \otimes V_2$ to $W_1 \otimes W_2$ given by the equation

$$(\varphi_1 \otimes \psi)(v_1 \otimes v_2) = \varphi_1(v_1) \otimes \varphi_2(v_2).$$

Tensor product of representations

Let $\varphi_1 \colon \mathrm{U}_q(\mathcal{L}(\mathfrak{g})) \to \mathrm{End}(V_1)$ and $\varphi_2 \colon \mathrm{U}_q(\mathcal{L}(\mathfrak{g})) \to V_2$ be representations of $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$. We define the tensor product of φ_1 and φ_2 as a representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$ on $V_1 \otimes V_2$ given by the equation

$$\varphi_1 \otimes_\Delta \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

- Using the properties of the universal $\mathcal{R}$-matrix one can demonstrate that

$$\mathcal{Y}_{\varphi_1 \otimes_\Delta \varphi_2} = \mathcal{Y}_{\varphi_1} \mathcal{Y}_{\varphi_2}$$

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Application to integrable systems

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- Assume now that we managed to prove that

$$\varphi_1 \otimes_\Delta \varphi_2 \simeq \chi_1 \oplus \cdots \oplus \chi_M.$$

- It follows that

$$\mathcal{Y}_{\varphi_1 \otimes_\Delta \varphi_2} = \mathcal{Y}_{\chi_1 \oplus \cdots \oplus \chi_M} = \mathcal{Y}_{\chi_1} + \cdots + \mathcal{Y}_{\chi_M}.$$

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we come to the equation

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Conclusion

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More details can be found in the papers

- A. V. Razumov, ℓ-weights and factorization of transfer operators, Theoretical and Mathematical Physics. **208** (2021) 1116–1143
- A. V. Razumov, Quantum groups and functional relations for arbitrary rank, Nuclear Physics, **B971** (2021) 115517 (51pp)

Thank you!

Thank you!

Questions?