GEOMETRY AND DYNAMICS OF THE BRANE-WORLD*

Roy Maartens†
Relativity and Cosmology Group, Portsmouth University, Portsmouth PO1 2EG, Britain

Recent developments in string theory have led to 5-dimensional warped spacetime models in which standard-model fields are confined to a 3-brane (the observed universe), while gravity can propagate in the fifth dimension. Gravity is localized near the brane at low energies, even if the extra dimension is noncompact. A review is given of the classical geometry and dynamics of these brane-world models. The field equations on the brane modify the general relativity equations in two ways: local 5-D effects are imprinted on the brane as a result of its embedding, and are significant at high energies; nonlocal effects arise from the 5-D Weyl tensor. The Weyl tensor transmits tidal (Coulomb), gravitomagnetic and gravitational wave effects to the brane from the 5-D nonlocal gravitational field. Local high-energy effects modify the dynamics of inflation, and increase the amplitude of scalar and tensor perturbations generated by inflation. Nonlocal effects introduce new features in cosmological perturbations. They induce a non-adiabatic mode in scalar perturbations and massive modes in vector and tensor perturbations, and they can support vector perturbations even in the absence of matter vorticity. In astrophysics, local and nonlocal effects introduce fundamental changes to gravitational collapse and black hole solutions.

I. INTRODUCTION

At high enough energies, Einstein's theory of general relativity breaks down and is likely to be a limit of a more general theory. In string theory/ M theory, gravity is a truly higher-dimensional theory, becoming effectively 4-dimensional at lower energies. Recent developments may offer a promising road towards a quantum gravity theory [1]. In brane-world models inspired by string/M theory, the standard-model fields are confined to a 3-brane, while the gravitational field can propagate in $3+d$ dimensions (the 'bulk'). The d extra dimensions need not all be small, or even compact: recently Randall and Sundrum [2] have shown that for $d=1$, gravity can be localized on a single 3-brane even when the fifth dimension is infinite. This noncompact localization arises via the exponential 'warp' factor in the non-factorizable metric:

$$d\tilde{s}^{\,2} = \exp(-2|y|/\ell)\left[-dt^2 + d\vec{x}^{\,2}\right] + dy^2\,. \tag{1}$$

For $y \neq 0$, this metric satisfies the 5-dimensional Einstein equations with negative 5-dimensional cosmological constant, $\widetilde{\Lambda} \propto -\ell^{-2}$. The brane is located at $y=0$, and the induced metric on the brane is a Minkowski metric. The bulk is a 5-dimensional anti-de Sitter metric, with $y=0$ as boundary, so that $y<0$ is identified with $y>0$, reflecting the Z_2 symmetry, with the brane as fixed point, that arises in string theory.

Perturbation of the metric (1) shows that the Newtonian gravitational potential on the brane is recovered at lowest order:

$$V(r) = \frac{GM}{r}\left(1 + \frac{2\ell^2}{3r^2}\right) + \cdots \tag{2}$$

Thus 4-dimensional gravity is recovered at low energies, with a first-order correction that is constrained by current sub-millimetre experiments [3]. The lowest order term corresponds to the massless graviton mode, bound to the brane, while the corrections arise from massive Kaluza-Klein modes in the bulk. Generalizing the Randall-Sundrum model to allow for matter on the brane leads to a generalization of the metric (1), and to a breaking of conformal flatness, since matter on the brane in general induces Weyl curvature in the bulk. Indeed, the massive Kaluza-Klein modes that produce the corrective terms in Eq. (2) reflect the bulk Weyl curvature that arises from a matter source on the brane.

At a classical level, the brane-world models have a rich geometrical structure, in which the bulk Weyl curvature tensor and its interaction with the curvature of the brane play an important role. The classical geometry and dynamics

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†roy.maartens@port.ac.uk

of brane-world models are important because they provide a foundation for testing the astrophysical and cosmological predictions of these models against observations. Any modification of Einstein's theory that arises in the quest for quantum gravity needs to pass the tests provided by increasingly accurate observations. Observers are necessarily bound to the brane, unable to access the bulk. Bulk effects are felt indirectly via their imprint on the brane geometry and dynamics. A covariant Lagrangian approach to the brane-world [4] brings out clearly the role of the Weyl tensors, the kinematical and dynamical quantities, and their relation to observations.

II. FIELD EQUATIONS ON THE BRANE

Instead of starting from a metric ansatz in special coordinates, one can generalize the Randall-Sundrum model by a covariant geometric approach, given by Shiromizu, Maeda and Sasaki [5]. (See also [6] for related geometrical results). The unit normal to the brane n^A defines the induced metric on the brane (and on all hypersurfaces orthogonal to n^A),

$$g_{AB} = \widetilde{g}_{AB} - n_A n_B \,, \tag{3}$$

where we use tildes to denote the 5-dimensional generalization of standard general relativity quantities. Without loss of generality, we can take n^A to be geodesic. A natural choice of coordinates is $x^A = (x^\mu, y)$, where $x^\mu = (t, x^i)$ are spacetime coordinates on the brane and $n_A = \delta_A{}^y$. The extrinsic curvature orthogonal to n^A is

$$K_{AB} = \tfrac{1}{2}\mathcal{L}_n g_{AB} = g_A{}^C \widetilde{\nabla}_C n_B \,, \tag{4}$$

so that $K_{[AB]} = 0 = K_{AB}n^B$, where square brackets denote anti-symmetrization and $\mathcal{L}$ is the Lie derivative.

The Gauss equation gives the 4-dimensional curvature tensor as

$$R_{ABCD} = \widetilde{R}_{EFGH}\, g_A{}^E g_B{}^F g_C{}^G g_D{}^H + 2K_{A[C}K_{D]B} \,, \tag{5}$$

and the Codazzi (or Mainardi-Codazzi) equation determines the change of K_{AB}:

$$\nabla_B K^B{}_A - \nabla_A K_B{}^B = \widetilde{R}_{BC}\, g_A{}^B n^C \,. \tag{6}$$

The 5-dimensional Einstein equations are

$$\widetilde{G}_{AB} = \widetilde{\kappa}^2 \left[-\widetilde{\Lambda}\widetilde{g}_{AB} + \delta(y)\left\{ -\lambda g_{AB} + T_{AB} \right\} \right] \,, \tag{7}$$

where $\widetilde{\kappa}^2 = 8\pi/\widetilde{M}_{\rm p}^3$, with $\widetilde{M}_{\rm p}$ the fundamental 5-dimensional Planck mass, which is typically much less than the effective Planck mass on the brane, $M_{\rm p} = 1.2 \times 10^{19}$ GeV. The brane tension is λ, and fields confined to the brane make up the brane energy-momentum tensor T_{AB}, with $T_{AB}n^B = 0$. The Randall-Sundrum model may be further generalized by allowing for a scalar field in the bulk [7].

Using Eqs. (5) and (7), it follows that [5]

$$\begin{aligned} G_{AB} = {} & -\tfrac{1}{2}\widetilde{\kappa}^2\widetilde{\Lambda} g_{AB} + K_C{}^C K_{AB} - K_A{}^C K_{CB} \\ & + \tfrac{1}{2}\left[K^{CD}K_{CD} - \left(K_C{}^C\right)^2 \right] g_{AB} - \mathcal{E}_{AB} \,, \end{aligned} \tag{8}$$

where

$$\mathcal{E}_{AB} = \widetilde{C}_{ACBD}\, n^C n^D \,, \tag{9}$$

is the projection of the bulk Weyl tensor orthogonal to n^A, with $\mathcal{E}_{[AB]} = 0 = \mathcal{E}_A{}^A$. Evaluating Eq. (8) on the brane (strictly, as $y \to \pm 0$) will give the field equations. First, we need to determine K_{AB} at the brane. The junction conditions across the brane imply that g_{AB} is continuous, while K_{AB} undergoes a jump due to the energy-momentum on the brane:

$$K^+_{AB} - K^-_{AB} = -\widetilde{\kappa}^2 \left[T_{AB} + \tfrac{1}{3}\left(\lambda - T_C{}^C\right) g_{AB} \right] \,. \tag{10}$$

The Z_2 symmetry implies that

$$K^-_{AB} = -K^+_{AB}\,, \tag{11}$$

and then

$$K_{AB} = -\tfrac{1}{2}\widetilde{\kappa}^2\left[T_{AB} + \tfrac{1}{3}\left(\lambda - T_C{}^C\right)g_{AB}\right]\,, \tag{12}$$

where we have dropped the (+) and we evaluate quantities on the brane by taking the limit $y \to +0$.

Finally we arrive at the induced field equations on the brane [5]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \widetilde{\kappa}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu}\,, \tag{13}$$

where $\kappa^2 = 8\pi/M_{\rm p}^2$. The energy scales are related to each other via

$$\lambda = 6\frac{\kappa^2}{\widetilde{\kappa}^4}\,, \quad \Lambda = \tfrac{1}{2}\widetilde{\kappa}^2\left(\widetilde{\Lambda} + \tfrac{1}{6}\widetilde{\kappa}^2\lambda^2\right)\,. \tag{14}$$

The higher-dimensional modifications of the standard Einstein equations on the brane are of two forms: first, the matter fields contribute local quadratic energy-momentum corrections via the tensor $S_{\mu\nu}$, which arise from the extrinsic curvature, and second, there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection $\mathcal{E}_{\mu\nu}$ of the bulk Weyl tensor. The local corrections are given by

$$S_{\mu\nu} = \tfrac{1}{12}T_\alpha{}^\alpha T_{\mu\nu} - \tfrac{1}{4}T_{\mu\alpha}T^\alpha{}_\nu + \tfrac{1}{24}g_{\mu\nu}\left[3T_{\alpha\beta}T^{\alpha\beta} - \left(T_\alpha{}^\alpha\right)^2\right]\,. \tag{15}$$

The Weyl tensor $\widetilde{C}_{ABCD}$ represents the free, nonlocal gravitational field in the bulk. The local part of the bulk gravitational field is the Einstein tensor $\widetilde{G}_{AB}$, which is determined locally via the bulk field equations (7). Thus *$\mathcal{E}_{\mu\nu}$ transmits nonlocal gravitational degrees of freedom from the bulk to the brane, including tidal (or Coulomb), gravito-magnetic and transverse traceless (gravitational wave) effects.*

There may be other branes in the bulk. Branes interact gravitationally via any Weyl curvature that they generate. On the observer's brane at $y = 0$, the presence of other branes is felt indirectly through their contribution to $\mathcal{E}_{\mu\nu}$.

As a consequence of the Codazzi equation (6), the form of the bulk energy-momentum tensor (which means that $\widetilde{R}_{BC}\,g_A{}^B n^C = 0$) and Z_2 symmetry, it follows that the brane energy-momentum tensor is conserved:

$$\nabla^\nu T_{\mu\nu} = 0\,. \tag{16}$$

When there are scalar and other fields in the bulk, this is no longer in general true [7,8], and non-conservation of $T_{\mu\nu}$ reflects an exchange of energy-momentum between the brane and the bulk. In the case when there is only a cosmological constant in the bulk, there is no such exchange. Using Eq. (16) in Eq. (13), the contracted Bianchi identities on the brane, $\nabla^\mu G_{\mu\nu} = 0$, imply that the projected Weyl tensor obeys the constraint

$$\nabla^\mu \mathcal{E}_{\mu\nu} = \frac{6\kappa^2}{\lambda}\,\nabla^\mu S_{\mu\nu}\,. \tag{17}$$

This shows that $\mathcal{E}_{\mu\nu}$ is sourced by energy-momentum terms, which in general include spatial gradients and time derivatives. Thus evolution and inhomogeneity in the matter fields can generate nonlocal gravitational effects in the bulk, which then 'backreact' on the brane.

The dynamical equations on the brane are equations (13), (16) and (17). It is important to note that these equations are not in general closed on the brane [5], since Eq. (17) does not determine $\mathcal{E}_{\mu\nu}$ in general, as further discussed below. This reflects the fact that there are bulk degrees of freedom which cannot be predicted from data available on the brane, for example, incoming gravitational radiation which impinges on the brane. One needs to solve the field equations in the bulk in order to fully determine $\mathcal{E}_{\mu\nu}$ on the brane.

III. COVARIANT INTERPRETATION OF GRAVITY LOCALIZATION

In the Randall-Sundrum model, with a flat brane, localization of gravity at the brane is understood perturbatively via Eq. (2). For a general matter distribution on a curved brane, we can provide a qualitative non-perturbative and covariant interpretation of gravity localization via tidal acceleration.

Consider a field of observers on the brane (e.g., observers comoving with matter) with 4-velocity u^μ, and let u^A be an extension off the brane (the result does not depend on the extension), so that $u^A n_A = 0$, $u^A u_A = -1$. The tidal acceleration in the n^A direction measured by the observers is $-n_A\widetilde{R}^A{}_{BCD}u^B n^C u^D$. Now

$$\widetilde{R}_{ABCD} = \widetilde{C}_{ABCD} + \tfrac{2}{3}\left\{\widetilde{g}_{A[C}\widetilde{R}_{D]B} + \widetilde{g}_{B[D}\widetilde{R}_{C]A}\right\} - \tfrac{1}{6}\widetilde{R}\widetilde{g}_{A[C}\widetilde{g}_{D]B}\,, \tag{18}$$

so that by the field equation (7) (and recalling that $T_{AB}n^B = 0$),

$$-\widetilde{R}_{ABCD}n^A u^B n^C u^D = -\mathcal{E}_{AB}u^A u^B + \tfrac{1}{6}\widetilde{\kappa}^2\widetilde{\Lambda}\,. \tag{19}$$

Taking the limit $y \to +0$, we get [4]

$$\text{tidal acceleration in off-brane direction } = \tfrac{1}{6}\widetilde{\kappa}^2\widetilde{\Lambda} - \mathcal{E}_{\mu\nu}u^\mu u^\nu\,. \tag{20}$$

Since $\widetilde{\Lambda} < 0$, it contributes to acceleration *towards* the brane. This reflects the confining role of the negative bulk cosmological constant on the gravitational field in the generalized Randall-Sundrum type models. Equation (20) also shows that localization of the gravitational field near the brane is enhanced if $\mathcal{E}_{\mu\nu}u^\mu u^\nu > 0$, which corresponds to a negative effective energy density on the brane from nonlocal bulk effects.

IV. COVARIANT DECOMPOSITION OF LOCAL AND NONLOCAL BULK EFFECTS

The general form of the brane energy-momentum tensor for any matter fields (scalar fields, perfect fluids, kinetic gases, dissipative fluids, etc.), including a combination of different fields, can be covariantly given as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \pi_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu\,. \tag{21}$$

Here ρ and p are the energy density and isotropic pressure, and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to u^μ on the brane. The energy flux obeys $q_\mu = q_{\langle\mu\rangle}$, and the anisotropic stress obeys $\pi_{\mu\nu} = \pi_{\langle\mu\nu\rangle}$, where angled brackets denote the projected, symmetric and tracefree part:

$$V_{\langle\mu\rangle} = h_\mu{}^\nu V_\nu\,, \quad W_{\langle\mu\nu\rangle} = \left[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta - \tfrac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right]W_{\alpha\beta}\,, \tag{22}$$

with round brackets denoting symmetrization. In an inertial frame at any point on the brane, we have $u^\mu = \delta^\mu{}_0$ and $h_{\mu\nu} = \mathrm{diag}(0,1,1,1)$, $q_\mu = (0, q_i)$, $\pi_{\mu 0} = 0$.

The tensor $S_{\mu\nu}$, which carries local bulk effects onto the brane, may then be irreducibly decomposed as

$$\begin{aligned}S_{\mu\nu} = {}& \tfrac{1}{24}\left[2\rho^2 - 3\pi_{\alpha\beta}\pi^{\alpha\beta}\right]u_\mu u_\nu + \tfrac{1}{24}\left[2\rho^2 + 4\rho p + \pi_{\alpha\beta}\pi^{\alpha\beta} - 4q_\alpha q^\alpha\right]h_{\mu\nu}\\ &- \tfrac{1}{12}(\rho + 2p)\pi_{\mu\nu} + \pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha + q_{\langle\mu}q_{\nu\rangle} + \tfrac{1}{3}\rho q_{(\mu}u_{\nu)} - \tfrac{1}{12}q^\alpha\pi_{\alpha(\mu}u_{\nu)}\,.\end{aligned} \tag{23}$$

This simplifies for a perfect fluid or minimally-coupled scalar field:

$$S_{\mu\nu} = \tfrac{1}{12}\rho^2 u_\mu u_\nu + \tfrac{1}{12}\rho\,(\rho + 2p)\,h_{\mu\nu}\,. \tag{24}$$

The quadratic energy-momentum corrections to standard general relativity will thus be significant for $\widetilde{\kappa}^4\rho^2 > 12\kappa^2\rho$, i.e., in the high-energy regime

$$\rho > \lambda \sim \left(\frac{\widetilde{M}_{\rm p}}{M_{\rm p}}\right)^2 \widetilde{M}_{\rm p}^4\,. \tag{25}$$

The lower bound arising from current tests for deviations from Newton's law is [9]

$$\widetilde{M}_{\rm p} > 10^5 \text{ TeV}\,, \quad \lambda^{1/4} > 100 \text{ GeV}\,. \tag{26}$$

(A much weaker limit is imposed by nucleosynthesis constraints.)

Nonlocal effects from the bulk are encoded in the brane tensor $\mathcal{E}_{\mu\nu}$, which can be decomposed as

$$\mathcal{E}_{\mu\nu} = -\frac{6}{\kappa^2\lambda}\left[\mathcal{U}\left(u_\mu u_\nu + \tfrac{1}{3}h_{\mu\nu}\right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu\right]\,. \tag{27}$$

The factor $6/\kappa^2\lambda$, which is $(\widetilde{\kappa}/\kappa)^4$ by Eq. (14), is introduced for dimensional reasons, and also since it ensures that in the general relativity limit, $\lambda^{-1} \to 0$, we have $\mathcal{E}_{\mu\nu} \to 0$. We have written $\mathcal{E}_{\mu\nu}$ as an effective energy-momentum tensor: the bulk Weyl tensor imprints on the brane an effective energy density, stresses and energy flux.

The effective nonlocal energy density on the brane, arising from the free gravitational field in the bulk, is

$$\mathcal{U} = -(\tfrac{1}{6}\kappa^2\lambda)\,\mathcal{E}_{\mu\nu}u^\mu u^\nu\,.$$

This nonlocal energy density need not be positive [10–12]. Since $\mathcal{E}_{\mu\nu}$ is tracefree, the effective nonlocal pressure is $\frac{1}{3}\mathcal{U}$. There is an effective nonlocal anisotropic stress

$$\mathcal{P}_{\mu\nu} = -(\tfrac{1}{6}\kappa^2\lambda)\,\mathcal{E}_{\langle\mu\nu\rangle}$$

on the brane, arising from the free gravitational field in the bulk, and

$$\mathcal{Q}_\mu = -(\tfrac{1}{6}\kappa^2\lambda)\,\mathcal{E}_{\langle\mu\rangle\nu}u^\nu$$

is an effective nonlocal energy flux on the brane, arising from the free gravitational field in the bulk.

If the bulk is anti-de Sitter (AdS_5), as in the Randall-Sundrum model, then $\mathcal{E}_{\mu\nu}=0$, since the bulk is conformally flat:

$$\text{AdS}_5 \text{ bulk: } \mathcal{E}_{\mu\nu} = 0\,. \tag{28}$$

The Randall-Sundrum model has a Minkowski brane, but AdS_5 can also admit a Friedmann brane, and satisfy the Einstein equations (7). However, the most general solution with a Friedmann brane is Schwarzschild-anti de Sitter spacetime [13]. Then it follows from the Friedmann symmetries that

$$\text{SAdS}_5 \text{ bulk, Friedmann brane: } \mathcal{Q}_\mu = 0 = \mathcal{P}_{\mu\nu}\,, \tag{29}$$

where $\mathcal{U}=0$ only if the mass of the black hole in the bulk is zero. The presence of the black hole leads to a 'dark radiation' term in the Friedmann equation (see below). For a static spherically symmetric brane (e.g. a static stellar interior or the exterior of a nonrotating black hole) [11]:

$$\text{static spherical brane: } \mathcal{Q}_\mu = 0\,. \tag{30}$$

This condition also holds for a Bianchi I brane [14].

The local and nonlocal bulk corrections may be consolidated into an effective total energy density, pressure, anisotropic stress and energy flux, since the modified field equations (13) take the standard Einstein form with a redefined energy-momentum tensor:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T^{\rm tot}_{\mu\nu}\,, \tag{31}$$

where

$$T^{\rm tot}_{\mu\nu} = T_{\mu\nu} + \frac{6}{\lambda}S_{\mu\nu} - \frac{1}{\kappa^2}\mathcal{E}_{\mu\nu}\,. \tag{32}$$

Then it follows from Eqs. (23) and (27) that

$$\rho^{\rm tot} = \rho + \frac{1}{4\lambda}\left(2\rho^2 - 3\pi_{\mu\nu}\pi^{\mu\nu}\right) + \frac{6}{\kappa^4\lambda}\mathcal{U} \tag{33}$$

$$p^{\rm tot} = p + \frac{1}{4\lambda}\left(2\rho^2 + 4\rho p + \pi_{\mu\nu}\pi^{\mu\nu} - 4q_\mu q^\mu\right) + \frac{2}{\kappa^4\lambda}\mathcal{U} \tag{34}$$

$$\pi^{\rm tot}_{\mu\nu} = \pi_{\mu\nu} + \frac{1}{2\lambda}\left[-(\rho+3p)\pi_{\mu\nu} + \pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^{\alpha} + q_{\langle\mu}q_{\nu\rangle}\right] + \frac{6}{\kappa^4\lambda}\mathcal{P}_{\mu\nu} \tag{35}$$

$$q^{\rm tot}_\mu = q_\mu + \frac{1}{4\lambda}\left(4\rho q_\mu - \pi_{\mu\nu}q^\nu\right) + \frac{6}{\kappa^4\lambda}\mathcal{Q}_\mu\,. \tag{36}$$

These general expressions make clear the local and nonlocal effects. They simplify in the case of a perfect fluid (or minimally coupled scalar field, or isotropic one-particle distribution function), i.e., for $q_\mu = 0 = \pi_{\mu\nu}$. However, we note that the total energy flux and anisotropic stress do not vanish in this case in general:

$$q^{\rm tot}_\mu = \frac{6}{\kappa^4\lambda}\mathcal{Q}_\mu\,, \quad \pi^{\rm tot}_{\mu\nu} = \frac{6}{\kappa^4\lambda}\mathcal{P}_{\mu\nu}\,. \tag{37}$$

Nonlocal bulk effects can contribute to effective imperfect fluid terms even when the matter on the brane has perfect fluid form.

V. LOCAL AND NONLOCAL CONSERVATION EQUATIONS

The brane energy-momentum tensor and the consolidated effective energy-momentum tensor are *both* conserved separately, by virtue of Eqs. (16) and (17). Conservation of $T_{\mu\nu}$ gives the standard general relativity energy and momentum conservation equations

$$\dot{\rho} + \Theta(\rho + p) + \mathrm{D}^\mu q_\mu + 2A^\mu q_\mu + \sigma^{\mu\nu}\pi_{\mu\nu} = 0\,, \tag{38}$$

$$\dot{q}_{\langle\mu\rangle} + \tfrac{4}{3}\Theta q_\mu + \mathrm{D}_\mu p + (\rho+p)A_\mu + \mathrm{D}^\nu \pi_{\mu\nu} + A^\nu \pi_{\mu\nu} + \sigma_{\mu\nu}q^\nu - [\omega, q]_\mu = 0\,. \tag{39}$$

In these equations, an overdot denotes $u^\nu\nabla_\nu$, $\Theta = \nabla^\mu u_\mu$ is the volume expansion rate of the u^μ congruence, $A_\mu = \dot{u}_\mu = A_{\langle\mu\rangle}$ is its 4-acceleration, $\sigma_{\mu\nu} = \mathrm{D}_{\langle\mu}u_{\nu\rangle}$ is its shear rate, and $\omega_\mu = -\frac{1}{2}\mathrm{curl}\, u_\mu = \omega_{\langle\mu\rangle}$ is its vorticity rate. On a Friedmann brane, $A_\mu = \omega_\mu = \sigma_{\mu\nu} = 0$ and $\Theta = 3H$, where $H = \dot{a}/a$ is the Hubble rate.

The covariant spatial curl is given by [15]

$$\mathrm{curl}\, V_\mu = \varepsilon_{\mu\alpha\beta}\mathrm{D}^\alpha V^\beta\,, \quad \mathrm{curl}\, W_{\mu\nu} = \varepsilon_{\alpha\beta(\mu}\mathrm{D}^\alpha W^\beta{}_{\nu)}\,, \tag{40}$$

where $\varepsilon_{\mu\nu\sigma}$ is the projection orthogonal to u^μ of the brane alternating tensor, and D_μ is the projected part of the brane covariant derivative, defined by

$$\mathrm{D}_\mu F^{\alpha\cdots}{}_{\cdots\beta} = (\nabla_\mu F^{\alpha\cdots}{}_{\cdots\beta})_\perp = h_\mu{}^\nu h^\alpha{}_\gamma \cdots h_\beta{}^\delta \nabla_\nu F^{\gamma\cdots}{}_{\cdots\delta}\,. \tag{41}$$

The covariant cross-product is $[V,Y]_\mu = \varepsilon_{\mu\alpha\beta}V^\alpha Y^\beta$. In a local inertial frame at a point on the brane, with $u^\mu = \delta^\mu{}_0$, we have: $0 = A_0 = \omega_0 = \sigma_{0\mu} = \varepsilon_{0\mu\nu} = [V,Y]_0 = \mathrm{curl}\, V_0 = \mathrm{curl}\, W_{0\mu}$ and $\mathrm{D}_\mu F^{\alpha\cdots}{}_{\cdots\beta} = \delta_\mu{}^i\delta^\alpha{}_j\cdots\delta_\beta{}^k\nabla_i F^{j\cdots}{}_{\cdots k}$.

The conservation of $T^{\rm tot}_{\mu\nu}$ gives, upon using Eqs. (33)–(39), what we can call nonlocal conservation equations [4]. The nonlocal energy conservation equation is a propagation equation for $\mathcal{U}$:

$$\begin{aligned}&\dot{\mathcal{U}} + \tfrac{4}{3}\Theta\mathcal{U} + \mathrm{D}^\mu\mathcal{Q}_\mu + 2A^\mu\mathcal{Q}_\mu + \sigma^{\mu\nu}\mathcal{P}_{\mu\nu}\\ &= \tfrac{1}{24}\kappa^4\left[6\pi^{\mu\nu}\dot{\pi}_{\mu\nu} + 6(\rho+p)\sigma^{\mu\nu}\pi_{\mu\nu} + 2\Theta\left(2q^\mu q_\mu + \pi^{\mu\nu}\pi_{\mu\nu}\right) + 2A^\mu q^\nu\pi_{\mu\nu}\right.\\ &\left.- 4q^\mu\mathrm{D}_\mu\rho + q^\mu\mathrm{D}^\nu\pi_{\mu\nu} + \pi^{\mu\nu}\mathrm{D}_\mu q_\nu - 2\sigma^{\mu\nu}\pi_{\alpha\mu}\pi_\nu{}^\alpha - 2\sigma^{\mu\nu}q_\mu q_\nu\right]\,.\end{aligned} \tag{42}$$

The nonlocal momentum conservation equation is a propagation equation for $\mathcal{Q}_\mu$:

$$\begin{aligned}&\dot{\mathcal{Q}}_{\langle\mu\rangle} + \tfrac{4}{3}\Theta\mathcal{Q}_\mu + \tfrac{1}{3}\mathrm{D}_\mu\mathcal{U} + \tfrac{4}{3}\mathcal{U}A_\mu + \mathrm{D}^\nu\mathcal{P}_{\mu\nu} + A^\nu\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}\mathcal{Q}^\nu - [\omega,\mathcal{Q}]_\mu\\ &= \tfrac{1}{24}\kappa^4\left[-4(\rho+p)\mathrm{D}_\mu\rho + 6(\rho+p)\mathrm{D}^\nu\pi_{\mu\nu} + q^\nu\dot{\pi}_{\langle\mu\nu\rangle} + \pi_\mu{}^\nu\mathrm{D}_\nu(2\rho+5p)\right.\\ &- \tfrac{2}{3}\pi^{\alpha\beta}\left(\mathrm{D}_\mu\pi_{\alpha\beta} + 3\mathrm{D}_\alpha\pi_{\beta\mu}\right) - 3\pi_{\mu\alpha}\mathrm{D}_\beta\pi^{\alpha\beta} + \tfrac{28}{3}q^\nu\mathrm{D}_\mu q_\nu\\ &+ 4\rho A^\nu\pi_{\mu\nu} - 3\pi_{\mu\alpha}A_\beta\pi^{\alpha\beta} + \tfrac{8}{3}A_\mu\pi^{\alpha\beta}\pi_{\alpha\beta} - \pi_{\mu\alpha}\sigma^{\alpha\beta}q_\beta\\ &+ \sigma_{\mu\alpha}\pi^{\alpha\beta}q_\beta + \pi_{\mu\nu}[\omega,q]^\nu - \varepsilon_{\mu\alpha\beta}\omega^\alpha\pi^{\beta\nu}q_\nu + 4(\rho+p)\Theta q_\mu\\ &\left.+ 6q_\mu A^\nu q_\nu + \tfrac{14}{3}A_\mu q^\nu q_\nu + 4q_\nu\sigma^{\alpha\beta}\pi_{\alpha\beta}\right]\,.\end{aligned} \tag{43}$$

All of the matter source terms on the right of these two equations, except for the first term on the right of Eq. (43), are imperfect fluid terms, and most of these terms are quadratic in the imperfect quantities q_μ and $\pi_{\mu\nu}$. For perfect fluid matter, only the $\mathrm{D}_\mu\rho$ term on the right of Eq. (43) survives, but in realistic cosmological and astrophysical models, further terms will survive. For example, terms linear in $\pi_{\mu\nu}$ will carry the photon quadrupole in cosmology or the shear viscous stress in stellar models.

In general, the 4 independent equations determine 4 of the 9 independent components of $\mathcal{E}_{\mu\nu}$ on the brane. What is missing, is an evolution equation for $\mathcal{P}_{\mu\nu}$ (which has up to 5 independent components). Thus in general, the projection of the 5-dimensional field equations onto the brane, together with Z_2 matching, does not lead to a closed system. Nor could we expect this to be the case, since there are bulk degrees of freedom whose impact on the brane cannot be predicted by brane observers. Our decomposition of $\mathcal{E}_{\mu\nu}$ has shown that *the evolution of the nonlocal energy density and flux (carrying scalar and vector modes from bulk gravitons) is determined on the brane, while the evolution of the nonlocal anisotropic stress (carrying tensor, as well as scalar and vector, modes) is not.*

In special cases the missing equation does not matter. For example, if $\mathcal{P}_{\mu\nu} = 0$, as in the case of a Friedmann brane, then the evolution of $\mathcal{E}_{\mu\nu}$ is determined by Eqs. (42) and (43). If the brane is stationary (with Killing vector parallel to u^μ), then evolution equations are not needed for $\mathcal{E}_{\mu\nu}$. However, small perturbations of these special cases will immediately restore the problem of missing information.

If the matter on the brane has a perfect-fluid energy-momentum tensor, the local conservation equations (38) and (39) reduce to

$$\dot{\rho} + \Theta(\rho + p) = 0\,, \tag{44}$$

$$\mathrm{D}_\mu p + (\rho + p) A_\mu = 0\,, \tag{45}$$

while the nonlocal conservation equations (42) and (43) reduce to

$$\dot{\mathcal{U}} + \tfrac{4}{3}\Theta\mathcal{U} + \mathrm{D}^\mu \mathcal{Q}_\mu + 2A^\mu \mathcal{Q}_\mu + \sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0\,, \tag{46}$$

$$\begin{aligned} &\dot{\mathcal{Q}}_{\langle\mu\rangle} + \tfrac{4}{3}\Theta\mathcal{Q}_\mu + \tfrac{1}{3}\mathrm{D}_\mu\mathcal{U} + \tfrac{4}{3}\mathcal{U}A_\mu + \mathrm{D}^\nu\mathcal{P}_{\mu\nu} + A^\nu\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}\mathcal{Q}^\nu - [\omega, \mathcal{Q}]_\mu \\ &\quad = -\tfrac{1}{6}\kappa^4(\rho + p)\mathrm{D}_\mu\rho\,. \end{aligned} \tag{47}$$

Equation (47) shows that [5] *if $\mathcal{E}_{\mu\nu} = 0$ and the brane energy-momentum tensor has perfect fluid form, then the density ρ must be homogeneous.* The converse does not hold, i.e., homogeneous density does *not* in general imply vanishing $\mathcal{E}_{\mu\nu}$. A simple example is the Friedmann case: Eq. (47) is trivially satisfied, while Eq. (46) becomes

$$\dot{\mathcal{U}} + 4H\mathcal{U} = 0\,. \tag{48}$$

This equation has the 'dark radiation' solution

$$\mathcal{U} = \mathcal{U}_o \left(\frac{a_o}{a}\right)^4 . \tag{49}$$

If $\mathcal{E}_{\mu\nu} = 0$, then the field equations on the brane form a closed system. Thus *for perfect fluid branes with homogeneous density and $\mathcal{E}_{\mu\nu} = 0$, the brane field equations form a consistent closed system.* However, there is no guarantee that the resulting brane metric can be embedded in a regular bulk.

It also follows as a corollary that *inhomogeneous density requires nonzero $\mathcal{E}_{\mu\nu}$.* For example, stellar solutions on the brane necessarily have $\mathcal{E}_{\mu\nu} \neq 0$ in the stellar interior if it is non-uniform. Perturbed Friedmann models on the brane also must have $\mathcal{E}_{\mu\nu} \neq 0$. Thus a nonzero $\mathcal{E}_{\mu\nu}$ is inevitable in realistic astrophysical and cosmological models.

For a perfect fluid at very high energies, i.e., $\rho \gg \lambda$, and for which we can neglect $\mathcal{U}$ (e.g., in an inflating cosmology), Eqs. (33) and (34) show that [16]

$$w^{\rm tot} \equiv \frac{p^{\rm tot}}{\rho^{\rm tot}} \approx 2w + 1\,, \tag{50}$$

$$(c_{\rm s}^2)^{\rm tot} \equiv \frac{\dot{p}^{\rm tot}}{\dot{\rho}^{\rm tot}} \approx c_{\rm s}^2 + w + 1\,, \tag{51}$$

where $w = p/\rho$ and $c_{\rm s}^2 = \dot{p}/\dot{\rho}$. Thus *at very high energies on the brane, the effective equation of state and sound speed are stiffened.* This can have important consequences in the early universe and during gravitational collapse. For example, in a very high-energy radiation era, $w = \frac{1}{3}$, the effective cosmological equation of state is ultra-stiff: $w^{\rm tot} \approx \frac{5}{3}$. In late-stage gravitational collapse of pressureless matter, $w = 0$, the effective equation of state is stiff, $w^{\rm tot} \approx 1$, and the effective pressure is nonzero and dynamically important.

VI. GRAVITATIONAL COLLAPSE TO A BLACK HOLE ON THE BRANE

The dynamics of gravitational collapse on the brane is not yet properly understood, given the complications that are introduced by local high-energy effects and nonlocal effects. In general relativity, it is known that nonrotating collapse to a black hole leads uniquely to the Schwarzschild metric, with no 'hair', i.e., no trace of the dynamics of the collapse process. On the brane, this is no longer true. *The Schwarzschild metric cannot describe the final state of collapse* [17,18,11], since it does not incorporate the 5-dimensional behaviour of the gravitational potential in the strong-field regime (the metric is incompatible with massive Kaluza-Klein modes). The appropriate metric on the brane to describe the black hole final state is not known, but a non-perturbative exterior solution should have *nonzero* $\mathcal{E}_{\mu\nu}$ in order to be compatible with massive Kaluza-Klein modes in the strong-field regime. In the end-state of collapse, we expect a nonlocal field $\mathcal{E}_{\mu\nu}$ which goes to zero at large distances, recovering the Schwarzschild weak-field limit, but which grows at short range. Furthermore, $\mathcal{E}_{\mu\nu}$ may carry a Weyl 'fossil' record of the collapse process.

The known black hole solutions on the brane include the Schwarzschild solution, for which $\mathcal{E}_{\mu\nu} = 0$, but also other solutions with nonzero nonlocal effects. A vacuum on the brane satisfies the field equations

$$R_{\mu\nu} = -\mathcal{E}_{\mu\nu}\,, \quad R^{\mu}{}_{\mu} = 0 = \mathcal{E}^{\mu}{}_{\mu}\,, \quad \nabla^{\nu}\mathcal{E}_{\mu\nu} = 0\,. \tag{52}$$

It follows that *Einstein-Maxwell solutions in general relativity produce vacuum solutions on the brane, where the electromagnetic field is replaced by a nonlocal Weyl energy-momentum tensor* [11]. Examples are the Reissner-Nördstrom-like solution discussed below, and a Vaidya-like solution [19]. The equations (52) form a closed system on the brane in the stationary case, including the static spherical case, for which

$$\Theta = 0 = \omega_{\mu} = \sigma_{\mu\nu}\,, \quad \dot{\mathcal{U}} = 0 = \mathcal{Q}_{\mu} = \dot{\mathcal{P}}_{\mu\nu}\,. \tag{53}$$

The nonlocal conservation equations reduce to

$$\tfrac{1}{3}\mathrm{D}_{\mu}\mathcal{U} + \tfrac{4}{3}\mathcal{U}A_{\mu} + \mathrm{D}^{\nu}\mathcal{P}_{\mu\nu} + A^{\nu}\mathcal{P}_{\mu\nu} = 0\,, \tag{54}$$

and the general solution of this and the remaining brane field equations, with metric of the form,

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2\,, \tag{55}$$

is then [11]

$$F = 1 - \left(\frac{2M}{M_{\rm p}^2}\right)\frac{1}{r} + \left(\frac{q}{\widetilde{M}_{\rm p}^2}\right)\frac{1}{r^2}\,, \tag{56}$$

$$\mathcal{E}_{\mu\nu} = -\left(\frac{q}{\widetilde{M}_{\rm p}^2}\right)\frac{1}{r^4}\left[u_{\mu}u_{\nu} - 2r_{\mu}r_{\nu} + h_{\mu\nu}\right]\,, \tag{57}$$

where r_{μ} is a unit radial vector.

This solution has the form of the general relativity Reissner-Nordström solution, but there is *no electric field* on the brane. Instead, the nonlocal Coulomb effects imprinted by the bulk Weyl tensor have induced a *'tidal' charge* parameter q. This parameter depends on the mass M on the brane, i.e., $q = q(M)$, since this is the source of bulk Weyl field (leaving aside the more complicated case where there may be additional Weyl sources in the bulk). In order to preserve the spacelike nature of the singularity, we need $q < 0$. This is in accord with the intuitive idea that the tidal charge *strengthens* the gravitational field, since it arises from the source mass M on the brane. By contrast, in the Reissner-Nordström solution of general relativity, $q = +Q^2$ weakens the gravitational field. Negative tidal charge, $q < 0$, means that there is one horizon on the brane, outside the Schwarzschild horizon:

$$r_{+} = \frac{M}{M_{\rm p}^2}\left[1 + \sqrt{1 - q\frac{M_{\rm p}^4}{M^2\widetilde{M}_{\rm p}^2}}\right] > r_{\rm s}\,. \tag{58}$$

The tidal-charge black hole metric does not satisfy the far-field r^{-3} correction to the gravitational potential [2,18,20,10], as in Eq. (2), and therefore cannot describe the end-state of collapse. However, Eq. (56) shows the correct 5-dimensional behaviour of the potential ($\propto r^{-2}$) at short distances, so that the tidal-charge metric should be a good approximation in the strong-field regime. This is the regime where the Schwarzschild metric on the brane fails, whereas the Schwarzschild metric is a good approximation at large distances.

The bulk solution corresponding to the tidal-charge brane metric is not known, although numerical investigations have been performed [21]. For the Schwarzschild special case $q = 0$, the brane is the surface $z = 1$ in AdS$_5$, in conformal coordinates [17]:

$$d\widetilde{s}^{\,2} = \left(\frac{3}{\kappa^2\lambda}\right)\frac{1}{z^2}\left[ds^2 + dz^2\right]\,. \tag{59}$$

Here ds^2 is the 4-dimensional Schwarzschild metric, given by Eqs. (55) and (56) with $q = 0$. The singularity $r = 0$ is a line along the z-axis, so that this bulk metric describes a 'black string'. The 5-dimensional horizon is the surface $g_{tt} = 0$ in the bulk, which is $r = 2M/M_{\rm p}^2$. This bulk horizon is a sphere of radius $r = r_{\rm s}$ on each z =constant surface, so that it has a cylindrical shape in the z-direction. The bulk is singular at the AdS$_5$ horizon, and the black string horizon is unstable [17,22].

If the physically realistic black hole bulk metric has the form

$$d\widetilde{s}^{\,2} = -F(r,y)dt^2 + J(r,y)dr^2 + L(r,y)r^2 d\Omega^2 + dy^2\,, \tag{60}$$

in Gaussian normal coordinates, then the bulk horizon is the surface $F(r,y) = 0$. Perturbative studies [18,22] and exact lower-dimensional solutions [23] have been used to probe the shape of the bulk horizon, but a non-perturbative 5-D solution has not yet been found.

Clearly the black hole solution, and the collapse process that leads to it, have a far richer structure in the brane-world than in general relativity. Another intriguing issue is how Hawking radiation is modified by bulk effects [24].

VII. EXACT COSMOLOGICAL SOLUTIONS

By Eqs. (49) and (81), the generalized Friedmann equation on a spatially homogeneous and isotropic brane is [25,26]

$$H^2 = \tfrac{1}{3}\kappa^2\rho\left(1+\frac{\rho}{2\lambda}\right)+\tfrac{1}{3}\Lambda-\frac{K}{a^2}+\frac{2\mathcal{U}_o}{\kappa^2\lambda}\left(\frac{a_o}{a}\right)^4, \tag{61}$$

where $K=0,\pm1$. The nonlocal term that arises from bulk Coulomb effects (also called the dark radiation term) is strongly limited by nucleosynthesis [25,27]:

$$\frac{1}{\kappa^2\lambda}\left(\frac{\mathcal{U}}{\rho}\right)_{\rm nucl} < 0.005\,. \tag{62}$$

This means that the nonlocal term is always sub-dominant during the radiation era, and rapidly becomes negligible after the radiation era.

If we neglect the nonlocal term, and take $\Lambda=0=K$, then the Friedmann equation during very high-energy inflation and early radiation-domination becomes

$$H^2\approx\frac{\kappa^2}{6}\frac{\rho^2}{\lambda}\,. \tag{63}$$

The enhanced expansion rate, compared to general relativity (where $H\propto\sqrt{\rho}$), introduces important changes to the dynamics of the early universe [9,28,29] and leads to an *increase in the amplitude of scalar* [9] *and tensor* [30] *perturbations generated by inflation.* If $\rho\gg\lambda$ at the start of the radiation era, then the solution of the Friedmann equation gives $a\propto t^{1/4}$.

The generalized Raychaudhuri equation (70) (with $\Lambda=0=\mathcal{U}$) reduces to

$$\dot H + H^2 = -\tfrac{1}{6}\kappa^2\left[\rho+3p+(2\rho+3p)\frac{\rho}{\lambda}\right], \tag{64}$$

and shows that the condition for inflation on the brane ($\ddot a>0$, i.e. $\dot H+H^2>0$) is [9]

$$w < -\frac{1}{3}\left(\frac{2\rho+\lambda}{\rho+\lambda}\right). \tag{65}$$

As $\rho/\lambda\to0$, we recover the general relativity result, $w<-\frac{1}{3}$. But in the very high-energy regime, we find $w<-\frac{2}{3}$.

The dynamics of homogeneous but anisotropic Bianchi branes has also been investigated [14,29]. In particular, high-energy bulk effects can strongly alter the dynamics near the singularity, because matter can dominate over shear anisotropy, opposite to the case of general relativity [14].

The bulk metric for a flat Friedmann brane may be given explicitly [25]:

$$d\tilde s^{\,2} = -N(t,y)^2dt^2 + A(t,y)^2d\vec x^{\,2}+dy^2\,, \tag{66}$$

where $A(t,0)=a(t)$ and t is proper time on the brane, so that $N(t,0)=1$. The metric functions are

$$N=\frac{\dot A}{\dot a}\,, \tag{67}$$

$$A^2 = -\frac{\rho}{2\lambda}\left(2+\frac{\rho}{\lambda}\right)a^2-\frac{6\,\mathcal{U}_oa_o^4}{\lambda^2a^2}-\left(1+\frac{\rho}{\lambda}\right)a^2\sinh(2\mu|y|) + \left[\frac{1}{2}\left(2+\frac{2\rho}{\lambda}+\frac{\rho^2}{\lambda^2}\right)a^2+\frac{6\,\mathcal{U}_oa_o^4}{\lambda^2a^2}\right]\cosh(2\mu y)\,, \tag{68}$$

with $\mu=\kappa^3\sqrt{6/\lambda}$. This bulk metric is Schwarzschild-AdS$_5$ spacetime [13] (in Gaussian normal coordinates), with the mass parameter of the black hole in the bulk proportional to $\mathcal{U}_o$.

VIII. PROPAGATION AND CONSTRAINT EQUATIONS

Section V gave propagation equations for the local and nonlocal energy density $\mathcal{U}$ and flux $\mathcal{Q}_\mu$. The remaining covariant equations on the brane are the propagation and constraint equations for the kinematic quantities Θ, A_μ, ω_μ, $\sigma_{\mu\nu}$, and for the nonlocal gravitational field on the brane. The kinematic quantities govern the relative motion of neighbouring fundamental world-lines. The nonlocal gravitational field on the brane is given by the *brane* Weyl tensor $C_{\mu\nu\alpha\beta}$. This splits into the gravito-electric and gravito-magnetic fields on the brane:

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta = E_{\langle\mu\nu\rangle}\,, \quad H_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\alpha\beta}C^{\alpha\beta}{}_{\nu\gamma}u^\gamma = H_{\langle\mu\nu\rangle}\,, \tag{69}$$

where $E_{\mu\nu}$ must not be confused with $\mathcal{E}_{\mu\nu}$. The Ricci identity for u^μ and the Bianchi identities $\nabla^\beta C_{\mu\nu\alpha\beta} = \nabla_{[\mu}(-R_{\nu]\alpha} + \frac{1}{6}Rg_{\nu]\alpha})$ produce the fundamental evolution and constraint equations governing the above covariant quantities [31]. The field equations are incorporated via the algebraic replacement of the Ricci tensor $R_{\mu\nu}$ by the effective total energy-momentum tensor, according to Eq. (31). The brane equations are derived directly from the standard general relativity versions [32] by simply replacing the energy-momentum tensor terms $\rho,\ldots$ by $\rho^{\rm tot},\ldots$. For a perfect fluid or minimally-coupled scalar field, the general equations [4] reduce to the following.

Generalized Raychaudhuri equation (expansion propagation):

$$\begin{aligned}&\dot\Theta + \tfrac{1}{3}\Theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - 2\omega_\mu\omega^\mu - \mathrm{D}^\mu A_\mu + A_\mu A^\mu + \tfrac{1}{2}\kappa^2(\rho+3p) - \Lambda \\ &\quad = -\tfrac{1}{2}\kappa^2(2\rho+3p)\frac{\rho}{\lambda} - \frac{6}{\kappa^2\lambda}\mathcal{U}\,.\end{aligned} \tag{70}$$

Vorticity propagation:

$$\dot\omega_{\langle\mu\rangle} + \tfrac{2}{3}\Theta\omega_\mu + \tfrac{1}{2}\mathrm{curl}\,A_\mu - \sigma_{\mu\nu}\omega^\nu = 0\,. \tag{71}$$

Shear propagation:

$$\dot\sigma_{\langle\mu\nu\rangle} + \tfrac{2}{3}\Theta\sigma_{\mu\nu} + E_{\mu\nu} - \mathrm{D}_{\langle\mu}A_{\nu\rangle} + \sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^\alpha + \omega_{\langle\mu}\omega_{\nu\rangle} - A_{\langle\mu}A_{\nu\rangle} = \frac{3}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}\,. \tag{72}$$

Gravito-electric propagation (Maxwell-Weyl E-dot equation):

$$\begin{aligned}&\dot E_{\langle\mu\nu\rangle} + \Theta E_{\mu\nu} - \mathrm{curl}\,H_{\mu\nu} + \tfrac{1}{2}\kappa^2(\rho+p)\sigma_{\mu\nu} \\ &\quad - 2A^\alpha\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^\beta - 3\sigma_{\alpha\langle\mu}E_{\nu\rangle}{}^\alpha + \omega^\alpha\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^\beta \\ &\quad = -\tfrac{1}{2}\kappa^2(\rho+p)\frac{\rho}{\lambda}\sigma_{\mu\nu} - \frac{1}{\kappa^2\lambda}\Big[4\mathcal{U}\sigma_{\mu\nu} + 3\dot{\mathcal{P}}_{\langle\mu\nu\rangle} + \Theta\mathcal{P}_{\mu\nu} + 3\mathrm{D}_{\langle\mu}\mathcal{Q}_{\nu\rangle} \\ &\quad + 6A_{\langle\mu}\mathcal{Q}_{\nu\rangle} + 3\sigma^\alpha{}_{\langle\mu}\mathcal{P}_{\nu\rangle\alpha} + 3\omega^\alpha\varepsilon_{\alpha\beta(\mu}\mathcal{P}_{\nu)}{}^\beta\Big]\,.\end{aligned} \tag{73}$$

Gravito-magnetic propagation (Maxwell-Weyl H-dot equation):

$$\begin{aligned}&\dot H_{\langle\mu\nu\rangle} + \Theta H_{\mu\nu} + \mathrm{curl}\,E_{\mu\nu} - 3\sigma_{\alpha\langle\mu}H_{\nu\rangle}{}^\alpha + \omega^\alpha\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^\beta + 2A^\alpha\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^\beta \\ &\quad = \frac{3}{\kappa^2\lambda}\left[\mathrm{curl}\,\mathcal{P}_{\mu\nu} - 3\omega_{\langle\mu}\mathcal{Q}_{\nu\rangle} + \sigma^\alpha{}_{(\mu}\varepsilon_{\nu)\alpha\beta}\mathcal{Q}^\beta\right]\,.\end{aligned} \tag{74}$$

Vorticity constraint:

$$\mathrm{D}^\mu\omega_\mu - A^\mu\omega_\mu = 0\,. \tag{75}$$

Shear constraint:

$$\mathrm{D}^\nu\sigma_{\mu\nu} - \mathrm{curl}\,\omega_\mu - \tfrac{2}{3}\mathrm{D}_\mu\Theta + 2[\omega,A]_\mu = -\frac{6}{\kappa^2\lambda}\mathcal{Q}_\mu\,. \tag{76}$$

Gravito-magnetic constraint:

$$\mathrm{curl}\,\sigma_{\mu\nu} + \mathrm{D}_{\langle\mu}\omega_{\nu\rangle} - H_{\mu\nu} + 2A_{\langle\mu}\omega_{\nu\rangle} = 0\,. \tag{77}$$

Gravito-electric divergence (Maxwell-Weyl div-E equation) :

$$\begin{aligned} &\mathrm{D}^\nu E_{\mu\nu} - \tfrac{1}{3}\kappa^2 \mathrm{D}_\mu\rho - [\sigma, H]_\mu + 3H_{\mu\nu}\omega^\nu \\ &= \frac{\kappa^2\rho}{3\lambda}\mathrm{D}_\mu\rho + \frac{1}{\kappa^2\lambda}\left(2\mathrm{D}_\mu\mathcal{U} - 2\Theta\mathcal{Q}_\mu - 3\mathrm{D}^\nu\mathcal{P}_{\mu\nu} + 3\sigma_{\mu\nu}\mathcal{Q}^\nu - 9[\omega, \mathcal{Q}]_\mu\right), \end{aligned} \tag{78}$$

where the covariant tensor commutator is $[W, Z]_\mu = \varepsilon_{\mu\alpha\beta} W^\alpha{}_\gamma Z^{\beta\gamma}$.

Gravito-magnetic divergence (Maxwell-Weyl div-H equation):

$$\begin{aligned} &\mathrm{D}^\nu H_{\mu\nu} - \kappa^2(\rho+p)\omega_\mu + [\sigma, E]_\mu - 3E_{\mu\nu}\omega^\nu \\ &= \kappa^2(\rho+p)\frac{\rho}{\lambda}\omega_\mu + \frac{1}{\kappa^2\lambda}\left(8\mathcal{U}\omega_\mu - 3\mathrm{curl}\,\mathcal{Q}_\mu - 3[\sigma, \mathcal{P}]_\mu - 3\mathcal{P}_{\mu\nu}\omega^\nu\right). \end{aligned} \tag{79}$$

Gauss-Codazzi equations on the brane ($\omega_\mu = 0$) [14]:

$$R^\perp_{\langle\mu\nu\rangle} + \dot{\sigma}_{\langle\mu\nu\rangle} + \Theta\sigma_{\mu\nu} - \mathrm{D}_{\langle\mu}A_{\nu\rangle} - A_{\langle\mu}A_{\nu\rangle} = \frac{6}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}\,, \tag{80}$$

$$R^\perp + \tfrac{2}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 2\kappa^2\rho - 2\Lambda = \frac{\kappa^2\rho^2}{\lambda} + \frac{12}{\kappa^2\lambda}\mathcal{U}\,, \tag{81}$$

where $R^\perp_{\mu\nu}$ is the Ricci tensor for 3-surfaces orthogonal to u^μ on the brane and $R^\perp = h^{\mu\nu}R^\perp_{\mu\nu}$.

The standard 4-dimensional general relativity results are regained when $\lambda^{-1} \to 0$, which sets all right hand sides to zero in Eqs. (70)–(81). Together with Eqs. (44)–(47), *these equations govern the dynamics of the matter and gravitational fields on the brane, incorporating both the local (quadratic energy-momentum) and nonlocal (projected 5-D Weyl) effects from the bulk.* Local terms are proportional to ρ/λ, and are significant only at high energies. Nonlocal terms contain $\mathcal{U}$, $\mathcal{Q}_\mu$ and $\mathcal{P}_{\mu\nu}$, with the latter two quantities introducing imperfect fluid effects, even though the matter has perfect fluid form.

Bulk effects give rise to important new driving and source terms in the propagation and constraint equations. The vorticity propagation and constraint, and the gravito-magnetic constraint have no direct bulk effects, but all other equations do. Local and nonlocal energy density are driving terms in the expansion propagation. The spatial gradients of local and nonlocal energy density provide sources for the gravito-electric field. The nonlocal anisotropic stress is a driving term in the propagation of shear and the gravito-electric/ -magnetic fields, and the nonlocal energy flux is a source for shear and the gravito-magnetic field. The Maxwell-Weyl equations show in detail the contribution to the nonlocal gravito-electromagnetic field on the brane, i.e., $(E_{\mu\nu}, H_{\mu\nu})$, from the nonlocal 5-dimensional Weyl field in the bulk.

The system of propagation and constraint equations, i.e. Eqs. (44)–(47) and (70)–(81), is exact and nonlinear, applicable to both cosmological and astrophysical modelling, including strong-gravity effects. In the next section we will linearize the system in order to study cosmological perturbations on the brane. A different linearization scheme could be developed for studying compact objects.

In general the system of equations is not closed: there is no evolution equation for the nonlocal anisotropic stress $\mathcal{P}_{\mu\nu}$. As noted above, closure on the brane can arise in special cases, when the nonlocal conservation equations (42) and (43) are sufficient to determine $\mathcal{E}_{\mu\nu}$. These special cases include:

$\mathcal{P}_{\mu\nu} = 0$, as on a Friedmann brane;
$\mathcal{E}_{\mu\nu} = 0$ and $\mathrm{D}_\mu\rho = 0$, with perfect fluid matter, as in special cases of Friedmann and Bianchi branes;
stationary brane, with $\dot{\mathcal{E}}_{\mu\nu} = 0$, so that Eqs. (42) and (43) reduce to constraint equations;
branes with isotropic 3-Ricci curvature, $\omega_\mu = 0 = R^\perp_{\langle\mu\nu\rangle}$.

The last case follows from Eq. (80), which becomes an equation defining $\mathcal{P}_{\mu\nu}$ in terms of quantities already determined via other equations. Special cases are Friedmann and Bianchi I branes. In general, for anisotropic 3-Ricci curvature, Eq. (80) is an equation determining $R^\perp_{\langle\mu\nu\rangle}$ in terms of other quantities.

Although the special cases above can give consistent closure on the brane, there is no guarantee that the brane is embeddable in a regular bulk. This is the case for a Friedmann brane, whose symmetries imply (together with Z_2 matching) that the bulk is Schwarzschild-AdS$_5$ [13]. A Schwarzschild brane can be embedded in a 'black string' bulk metric, but this has singularities [17,22].

IX. COSMOLOGICAL PERTURBATIONS

The dynamics of the background homogeneous brane provides important constraints on brane-world cosmologies. Roughly speaking, these constraints amount to the statement that the brane-world should reproduce the standard Friedmann dynamics from nucleosynthesis onwards. However, much stricter constraints are implied by the growing body of data on the cosmic microwave background (CMB) radiation and large-scale structure (LSS), since this data indirectly probes earlier times. In order to confront brane-world models with the data, we need to study perturbations on the brane.

As is apparent from the exact nonlinear equations in the previous section, cosmological perturbations on the brane are qualitatively more complicated than in general relativity. Not only is the background more complicated, but effects from the bulk that are imprinted on the brane include degrees of freedom that cannot be predicted from data available to brane observers. A complete understanding of brane perturbations therefore necessarily involves the analysis of bulk perturbations – for which the mode equations are *partial* differential equations, with nontrivial boundary conditions.

A covariant brane-perturbation theory has been developed [4] and applied to large-scale density perturbations [16]. Metric-based general formalisms for bulk perturbations have been developed [33], and large-scale perturbations generated from quantum fluctuations during de Sitter inflation on the brane have also been partly computed [9,34,35,30,36]. Large-scale scalar perturbations and their impact on the CMB have been analysed [27].

The results so far may be summarized as follows:

Tensor perturbations: *Bulk effects produce a massless mode during inflation and a continuum of massive Kaluza-Klein modes* [35,30], *with* $m > \frac{3}{2}H$. The massive modes stay in the vacuum state, and on large scales there is a constant mode with enhanced amplitude (compared to general relativity) [30]. We expect no qualitative change on large scales to 4-dimensional general relativity tensor modes in the CMB and LSS, but there could be a significant change on small scales due to the massive modes.

Vector perturbations: *Bulk effects can support vector perturbations, even without matter vorticity* [4,36], but these modes, which do not arise in general relativity, are massive (there is no normalizable massless mode), and stay in the vacuum state during inflation on the brane [36]. The momentum can be determined on large scales without solving the bulk perturbations, but *the vector Sachs-Wolfe effect cannot be found on-brane* [36], because of the nonlocal anisotropic stress, which is undetermined on the brane. There are possibly qualitative changes to 4-dimensional general relativity vector modes in the CMB and LSS on large scales.

Scalar perturbations: *Bulk effects introduce a non-adiabatic mode on large scales* [4,16,27]. Density perturbations on large scales can be solved on-brane, without solving for the bulk perturbations [16], but *the Sachs-Wolfe effect cannot be found on-brane* [27], because of the nonlocal anisotropic stress, which is undetermined on the brane. There are possibly qualitative changes to 4-dimensional general relativity scalar modes in the CMB and LSS, even on large scales, and probably significant changes on small scales.

The quantitative results up to now are confined to large scales. Further progress requires the integration of the bulk mode equations. In the simplest case, for tensor perturbations on a flat Friedmann brane, the mode equation is [30]

$$\frac{\partial}{\partial t}\left(\frac{A^3}{N}\frac{\partial \mathcal{H}}{\partial t}\right)-\frac{\partial}{\partial y}\left(A^3 N\frac{\partial \mathcal{H}}{\partial y}\right)+k^2 AN\mathcal{H}=0\,, \tag{82}$$

where A and N are given by Eqs. (67) and (68). The perturbed metric, in Gaussian normal coordinates, is

$$d\tilde{s}^2=-N(t,y)^2dt^2+A(t,y)^2\left[\delta_{ij}+\mathcal{H}_{ij}(t,\vec{x},y)\right]dx^idx^j+dy^2\,, \tag{83}$$

where $\mathcal{H}_i{}^i=0=\partial^j\mathcal{H}_{ij}$ and

$$\mathcal{H}_{ij}(t,\vec{x},y)=\mathcal{H}(t,y)\exp(\mathrm{i}\vec{k}\cdot\vec{x})\,\mathrm{e}_{ij}\,, \tag{84}$$

with e_{ij} a polarization tensor. At the brane, the boundary condition is

$$\left(\frac{\partial\mathcal{H}}{\partial y}\right)_{y=0}=\pi^{\mathrm{T}}\,, \tag{85}$$

where $\pi^{\rm T}(t)\exp({\rm i}\vec{k}\cdot\vec{x})\,{\rm e}_{ij}$ is the anisotropic stress of matter on the brane.

In order to illustrate some of the features of perturbations on the brane, we can use the brane-based covariant approach [4]. The exact nonlinear equations in previous sections can be linearized as follows. The limiting case of the background Friedmann brane is characterized by the vanishing of all inhomogeneous and anisotropic quantities:

$$
{\rm D}_\mu f = V_\mu = W_{\mu\nu} = 0\,, \tag{86}
$$

where $f=\rho,p,\Theta,\mathcal{U}$, and $V_\mu=A_\mu,\omega_\mu,\mathcal{Q}_\mu$, and $W_{\mu\nu}=\sigma_{\mu\nu},E_{\mu\nu},H_{\mu\nu},\mathcal{P}_{\mu\nu}$. These quantities are then first-order of smallness in the linearization scheme, and since they vanish in the background, they are gauge-invariant [37]. The linearized conservation equations (assuming adiabatic matter perturbations) are

$$
\dot\rho+\Theta(\rho+p)=0\,, \tag{87}
$$

$$
c_{\rm s}^2{\rm D}_\mu\rho+(\rho+p)A_\mu=0\,, \tag{88}
$$

$$
\dot{\mathcal{U}}+\tfrac{4}{3}\Theta\mathcal{U}+{\rm D}^\mu\mathcal{Q}_\mu=0\,, \tag{89}
$$

$$
\dot{\mathcal{Q}}_\mu+4H\mathcal{Q}_\mu+\tfrac{1}{3}{\rm D}_\mu\mathcal{U}+\tfrac{4}{3}\mathcal{U}A_\mu+{\rm D}^\nu\mathcal{P}_{\mu\nu}=-\tfrac{1}{6}\kappa^4(\rho+p){\rm D}_\mu\rho\,. \tag{90}
$$

Linearization of the propagation and constraint equations leads to:

$$
\dot\Theta+\tfrac{1}{3}\Theta^2-{\rm D}^\mu A_\mu+\tfrac{1}{2}\kappa^2(\rho+3p)-\Lambda=-\tfrac{1}{2}\kappa^2(2\rho+3p)\frac{\rho}{\lambda}-\frac{6}{\kappa^2\lambda}\mathcal{U}\,, \tag{91}
$$

$$
\dot\omega_\mu+2H\omega_\mu+\tfrac{1}{2}{\rm curl}\,A_\mu=0\,, \tag{92}
$$

$$
\dot\sigma_{\mu\nu}+2H\sigma_{\mu\nu}+E_{\mu\nu}-{\rm D}_{\langle\mu}A_{\nu\rangle}=\frac{3}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}\,, \tag{93}
$$

$$
\begin{aligned}
&\dot E_{\mu\nu}+3HE_{\mu\nu}-{\rm curl}\,H_{\mu\nu}+\tfrac{1}{2}\kappa^2(\rho+p)\sigma_{\mu\nu}=-\tfrac{1}{2}\kappa^2(\rho+p)\frac{\rho}{\lambda}\sigma_{\mu\nu}\\
&\quad-\frac{1}{\kappa^2\lambda}\left[4\mathcal{U}\sigma_{\mu\nu}+3\dot{\mathcal{P}}_{\langle\mu\nu\rangle}+\Theta\mathcal{P}_{\mu\nu}+3{\rm D}_{\langle\mu}\mathcal{Q}_{\nu\rangle}\right]\,,
\end{aligned} \tag{94}
$$

$$
\dot H_{\mu\nu}+3HH_{\mu\nu}+{\rm curl}\,E_{\mu\nu}=\frac{3}{\kappa^2\lambda}{\rm curl}\,\mathcal{P}_{\mu\nu}\,, \tag{95}
$$

$$
{\rm D}^\mu\omega_\mu=0\,, \tag{96}
$$

$$
{\rm D}^\nu\sigma_{\mu\nu}-{\rm curl}\,\omega_\mu-\tfrac{2}{3}{\rm D}_\mu\Theta=-\frac{6}{\kappa^2\lambda}\mathcal{Q}_\mu\,, \tag{97}
$$

$$
{\rm curl}\,\sigma_{\mu\nu}+{\rm D}_{\langle\mu}\omega_{\nu\rangle}-H_{\mu\nu}=0\,, \tag{98}
$$

$$
{\rm D}^\nu E_{\mu\nu}-\tfrac{1}{3}\kappa^2{\rm D}_\mu\rho=\frac{\kappa^2\rho}{3\lambda}{\rm D}_\mu\rho+\frac{1}{\kappa^2\lambda}\left[2{\rm D}_\mu\mathcal{U}-2\Theta\mathcal{Q}_\mu-3{\rm D}^\nu\mathcal{P}_{\mu\nu}\right]\,, \tag{99}
$$

$$
{\rm D}^\nu H_{\mu\nu}-\kappa^2(\rho+p)\omega_\mu=\kappa^2(\rho+p)\frac{\rho}{\lambda}\omega_\mu+\frac{1}{\kappa^2\lambda}\left[8\mathcal{U}\omega_\mu-3{\rm curl}\,\mathcal{Q}_\mu\right]\,. \tag{100}
$$

Equations (87), (89) and (91) do not provide gauge-invariant equations for perturbed quantities, but their spatial gradients do.

A covariant, gauge-invariant (and purely local) splitting into scalar, vector and tensor modes is given by (compare [38])

$$
V_\mu={\rm D}_\mu V+\bar V_\mu\,, \tag{101}
$$

$$
W_{\mu\nu}={\rm D}_{\langle\mu}{\rm D}_{\nu\rangle}W+{\rm D}_{\langle\mu}\bar W_{\nu\rangle}+\bar W_{\mu\nu}\,, \tag{102}
$$

where an overbar denotes a transverse (divergence-free) quantity (note that $W_{\mu\nu}$ is already tracefree). Purely scalar modes are characterized by

$$
\bar V_\mu=\bar W_\mu=\bar W_{\mu\nu}=0\,, \tag{103}
$$

and standard identities [4], the vorticity constraint equation (96) and the gravito-magnetic constraint equation (98) then show that

$$
{\rm curl}\,V_\mu=0={\rm curl}\,W_{\mu\nu}\,,\ \ {\rm D}^\nu W_{\mu\nu}=\tfrac{2}{3}{\rm D}^2({\rm D}_\mu W)\,,\ \ \omega_\mu=0=H_{\mu\nu}\,. \tag{104}
$$

Vector modes obey

$$V_\mu = \bar{V}_\mu \,, \quad W_{\mu\nu} = \mathrm{D}_{\langle\mu}\bar{W}_{\nu\rangle} \,, \quad \mathrm{curl}\,\mathrm{D}_\mu f = -2\dot{f}\omega_\mu \,, \tag{105}$$

and it then follows that

$$\mathrm{D}^\nu W_{\mu\nu} = \tfrac{1}{2}\mathrm{D}^2\bar{W}_\mu \,, \quad \mathrm{curl}\,W_{\mu\nu} = \tfrac{1}{2}\mathrm{D}_{\langle\mu}\bar{W}_{\nu\rangle} \,. \tag{106}$$

Tensor modes are covariantly characterized by

$$\mathrm{D}_\mu f = 0 = V_\mu \,, \quad W_{\mu\nu} = \bar{W}_{\mu\nu} \,. \tag{107}$$

A. Density perturbations on the brane

We define the density and expansion scalars (as in general relativity [37])

$$\Delta = \frac{a^2}{\rho}\mathrm{D}^2\rho \,, \quad Z = a^2\mathrm{D}^2\Theta \,, \tag{108}$$

and scalars describing inhomogeneity in the nonlocal quantities:

$$U = \frac{a^2}{\rho}\mathrm{D}^2\mathcal{U} \,, \quad Q = \frac{a}{\rho}\mathrm{D}^2\mathcal{Q} \,, \quad P = \frac{1}{\rho}\mathrm{D}^2\mathcal{P} \,. \tag{109}$$

Then we take the comoving spatial Laplacian of Eqs. (87), (89) and (91), using standard identities [4]. This leads to a system of 4 evolution equations for Δ, Z, U, and Q. In general the system is under-determined, since there is no evolution equation for P. However, P only arises via its Laplacian, and on large scales, the *system closes and brane observers can predict density perturbations from initial conditions intrinsic to the brane.* The system on large scales is

$$\dot{\Delta} = 3wH\Delta - (1+w)Z \,, \tag{110}$$

$$\dot{Z} = -2HZ - \frac{6\rho}{\kappa^2\lambda}U - \tfrac{1}{2}\kappa^2\rho\left[1 + (4+3w)\frac{\rho}{\lambda} - \frac{12c_{\mathrm{s}}^2\mathcal{U}}{(1+w)\kappa^4\lambda\rho}\right]\Delta \,, \tag{111}$$

$$\dot{U} = (3w-1)HU - \left(\frac{4c_{\mathrm{s}}^2}{1+w}\right)\left(\frac{\mathcal{U}}{\rho}\right)H\Delta - \left(\frac{4\mathcal{U}}{3\rho}\right)Z \,, \tag{112}$$

$$\dot{Q} = (1-3w)HQ - \frac{1}{3a}U + \frac{1}{6a}\left[\frac{8c_{\mathrm{s}}^2\mathcal{U}}{(1+w)\rho} - \kappa^4\rho^2(1+w)\right]\Delta \,. \tag{113}$$

In standard general relativity, only the first two equations apply, with λ^{-1} set to zero in Eq. (111), so that we can decouple the density perturbations via a second-order equation for Δ, whose independent solutions are adiabatic growing and decaying modes. *Bulk effects introduce a new non-adiabatic mode:* there are 3 coupled equations in Δ, Z and U, plus a decoupled equation for Q, which is determined once the other 3 quantities are solved for. Some solutions are found in [16] for $\mathcal{U} = 0$ in the background, and they show how the fluctuations U in $\mathcal{U}$ introduce effective entropy perturbations, and how Δ evolves differently from the standard general relativity case when $\rho \gg \lambda$ (see also [27]).

B. Vector perturbations on the brane

The linearized vorticity propagation equation (92) does not carry any bulk effects, and vorticity decays with expansion as in standard general relativity, reflecting the fact that angular momentum conservation holds on the brane. The gravito-magnetic divergence equation (100), becomes

$$\mathrm{D}^2\bar{H}_\mu = 2\kappa^2(\rho+p)\left[1 + \frac{\rho}{\lambda}\right]\omega_\mu + \frac{2}{\kappa^2\lambda}\left[8\mathcal{U}\omega_\mu - 3\mathrm{curl}\,\bar{\mathcal{Q}}_\mu\right] \,, \tag{114}$$

on using Eq. (106). This shows that the local and nonlocal energy density and the nonlocal energy flux provide additional sources for the brane gravito-magnetic field. In standard general relativity, it is necessary to increase the angular momentum density $(\rho+p)\omega_\mu$ in order to increase the gravito-magnetic field, but bulk effects allow additional sources. In particular, unlike in general relativity, *it is possible to source vector perturbations even when the vorticity vanishes,* since curl $\bar{\mathcal{Q}}_\mu$ may be nonzero.

We can find a closed system of equations on the brane for ω_μ and curl $\bar{\mathcal{Q}}_\mu$, on large scales. Using Eqs. (90), (92), (105) and (106), we find

$$\dot{\bar{\alpha}}_\mu + \left(1-3c_{\rm s}^2\right) H\bar{\alpha}_\mu = 0\,, \tag{115}$$

$$\dot{\bar{\beta}}_\mu + (1-3w)H\bar{\beta}_\mu = \left[\frac{8}{3}H\left(3c_{\rm s}^2-1\right)\frac{\mathcal{U}}{\rho} - \kappa^4(1+w)^2\rho^2\right]\bar{\alpha}_\mu\,, \tag{116}$$

where

$$\bar{\alpha}_\mu = a\,\omega_\mu\,,\quad \bar{\beta}_\mu = \frac{a}{\rho}\text{curl}\,\mathcal{Q}_\mu\,, \tag{117}$$

are dimensionless vector perturbations. Thus we can solve for the matter and nonlocal vorticity on large scales, without solving the bulk perturbation equations, similar to the case of scalar perturbations. The matter vorticity sources nonlocal vorticity, but during slow-roll inflation this only leads to a highly damped mode $\propto a^{-8}$. In very high-energy radiation domination however, there is a weakly growing mode:

$$\bar{\alpha}_\mu = b_\mu\,,\quad \bar{\beta}_\mu = c_\mu + Bb_\mu \ln\left(\frac{a}{a_o}\right)\,, \tag{118}$$

where $\dot{B} = \dot{b}_\mu = \dot{c}_\mu = 0$.

C. Tensor perturbations on the brane

For the transverse traceless modes on the brane, the scalar equations reduce to background equations, the vector equations drop out and we can derive a covariant wave equation for the tensor shear from the remaining equations (93)–(95) and (98):

$$\begin{aligned}&\mathrm{D}^2\bar{\sigma}_{\mu\nu} - \ddot{\bar{\sigma}}_{\mu\nu} - 5H\dot{\bar{\sigma}}_{\mu\nu} - \left[2\Lambda + \tfrac{1}{2}\kappa^2(\rho-3p) - \tfrac{1}{2}\kappa^2(\rho+3p)\frac{\rho}{\lambda}\right]\bar{\sigma}_{\mu\nu}\\ &= -\frac{6}{\kappa^2\lambda}\left[\dot{\bar{\mathcal{P}}}_{\mu\nu} + 2H\bar{\mathcal{P}}_{\mu\nu}\right]\,.\end{aligned} \tag{119}$$

In standard general relativity, the right hand side falls away, and once $\bar{\sigma}_{\mu\nu}$ is determined, one can determine $\bar{E}_{\mu\nu}$ from Eq. (93) and $\bar{H}_{\mu\nu}$ from Eq. (98) (the shear is a gravito-electromagnetic potential). Nonlocal bulk effects provide driving terms that are like anisotropic stress terms in general relativity [39]. In the latter case however, the evolution of anisotropic stress is determined by the Boltzmann equation or other intrinsic physics. The nonlocal anisotropic stress $\bar{\mathcal{P}}_{\mu\nu}$ from the bulk Weyl tensor is not determined by brane equations, and can in principle introduce significant changes to the shear via Eq. (119).

X. CONCLUSION

By adopting a covariant approach based on physical and geometrical quantities that are in principle measurable by brane-observers, we have given a classical analysis of intrinsic cosmological and astrophysical dynamics in generalized Randall-Sundrum-type brane-worlds. We have emphasised the role of the Weyl tensors in the bulk and on the brane, and we have carefully delineated what can and can not be predicted by brane observers without additional information from the unobservable bulk. Intriguing new effects are introduced by the extra dimension, and there are exciting problems to be solved, especially on brane-world black holes and gravitational collapse, and on cosmological perturbations and the CMB and LSS.

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