Brane worlds

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Abstract. Brane worlds and large extra dimensions attract a lot of attention as possible new paradigms for spacetime. I review the theory of gravity on 3-branes with a focus on the codimension 1 models. However, for a new result it is also pointed out that the cosmological evolution of the 3-brane in the model of Dvali, Gabadadze and Porrati may follow the standard Friedmann equation.

Contents

1. Introduction

Observation tells us that the number of macroscopic degrees of freedom of a particle at presently accessible energies per particle $E < 1\,\mathrm{TeV}$ is three, corresponding to the three spatial dimensions which we encounter in our everyday lives. At the highest presently accessible energies in collider experiments this is confirmed through conservation of the quantum numbers of the tangential SO(3,1) symmetry, energy and momentum: Momentum along a translationally symmetric brane would be conserved anyhow in a particle scattering experiment, but the fact that we do not need to account for any additional transverse momenta in energy conservation shows that particles do not escape into any hidden dimensions at presently accessible energies.

Of course, the qualification with regard to the energy range also implies the well-known fact that small extra dimensions are well compatible with experimental evidence for three approximately flat macroscopic spatial dimensions‡, as has been pointed out for the first time in the case of small *periodic* extra dimensions [74, 75]: Shifting a particle into a periodic dimension of radius R requires an energy (for $mc \ll \hbar/R$)

$$\Delta E = \frac{\hbar c}{R}. \tag{1}$$

Beyond Kaluza–Klein, resolving small *non-periodic compact extra dimensions* is also energetically prohibited due to the uncertainty principle, whence a TeV-scale accelerator should be able to probe dimensions of size§ $R > 10^{-17}\,\mathrm{cm} \approx 10^{16}\ell_{\mathrm{Planck}}$, and *non-compact finite-volume theories of extra dimensions* provide phenomenologically acceptable generalizations of the Kaluza–Klein framework through the discreteness of the internal harmonic modes [111, 93, 57, 97].

These frameworks for extra dimensions with energetically suppressed Kaluza–Klein modes had attracted a lot of attention, partly for their own sake and partly for the need to include extra dimensions in string theory, see [7, 108, 61, 41, 97] and references there. Extra dimensions with TeV-scale Kaluza–Klein modes had also been discussed in string theory [5, 67, 113, 82, 44, 71].

By the same token, the estimate $\Delta x \approx \hbar c/E$ for the resolving power seems to rule out *large* extra dimensions for which the energy gap would become so small that it should show up in particle physics experiments as missing energy or through excitation of a first Kaluza–Klein level‖.

In spite of this apparent obstacle, a framework for phenomenologically acceptable large extra dimensions is evolving in the literature and has attracted a lot of attention since ~ 1998. There are two main themes in this subject: Matter must not escape into large extra dimensions, and any viable theory of large extra dimensions must produce a phenomenologically acceptable four-dimensional theory of gravity and cosmology.

‡ According to our current understanding our knowledge of the large scale structure of the universe is confined to the Hubble radius of order $\sim 10^{10}$ light years, if the early hot and dense phase that we see directly in the cosmic background radiation and indirectly in the success of the theory of primordial nucleosynthesis emerged from an initial singularity, or if the thermal properties of the hot and dense phase imply that any prior information has been erased. This qualification is understood in any statements about macroscopic properties of spacetime.

§ The highest energy single particle events that we seem to observe are the ultrahigh energy cosmic rays with energies reaching almost $10^{12}\,\mathrm{GeV}$. Our planet is hit by such a high-energetic cosmic ray roughly once per year per $100\,\mathrm{km}^2$ [107, 89, 65], and if the observed extremely high-energetic atmospheric jets are triggered by single particles, the propagation of these particles through spacetime would be affected by extra dimensions of size $R > 10^{-26}\,\mathrm{cm} \approx 10^{7}\ell_{\mathrm{Planck}}$.

‖ This remark entails a definition of "large extra dimension": An extra dimension is *large* if the corresponding Kaluza–Klein modes of matter fields could be generated in present day accelerators.

The gravity problem is even more relevant than the escape problem, because it is mathematically fully consistent to devise models where matter degrees of freedom are *a priori* bound to a four-dimensional submanifold of a higher-dimensional spacetime while the extra dimensions can only be probed by gravitons. In the case of one large extra dimension the latter type of models might be described by action principles

$$S = \int_{\mathcal{M}_{4,1}} d^5x\, \mathcal{L}_G + \int_{\mathcal{M}_{3,1}} d^4x\, \mathcal{L}_M, \tag{2}$$

where the Lagrangian $\mathcal{L}_G$ would comprise all the gravity-like degrees of freedom and $\mathcal{L}_M$ would comprise all the excitations which can only live on the $(3+1)$-dimensional submanifold $\mathcal{M}_{3,1}$ (see, however, Sec. 4 for the necessity to take into account extrinsic curvature terms on $\mathcal{M}_{3,1}$ and Sec. 5.3 for the possibility to add the intrinsic curvature of $\mathcal{M}_{3,1}$).

From a physical perspective, such a split of dynamical degrees of freedom with respect to the supporting manifold may seem counter-intuitive at first sight, but we should contemplate the possibility that nature may supply different degrees of freedom on different manifolds. However, from a slightly more conservative point of view, dynamical binding mechanisms of matter to a four-dimensional submanifold of a higher-dimensional spacetime have been proposed already in [3, 101, 109, 106]. In the models proposed by Akama and by Rubakov and Shaposhnikov trapping of matter to a submanifold is implemented through the coupling of matter to solitonic scalar fields: Akama had used a Nielsen–Olesen vortex in $5+1$ dimensions to attract matter to a 3-brane [3], while Rubakov and Shaposhnikov realized the matter attracting 3-brane as a domain wall in $4+1$ dimensions, see also [4, 100] for recent more general discussions of solitonic binding mechanisms. Motivated by the work on solitonic realizations, Visser had pointed out that matter might also be gravitationally bound to submanifolds. Visser specifically proposed a model where a $U(1)$ gauge field in $4+1$ dimensions induces a background metric which binds particles to a 3-brane orthogonal to the $U(1)$ gauge field [109], and later Squires pointed out that this effect can also be due to a bulk gravitational constant [106].

The model of Dvali, Gabadadze and Porrati [47] to be discussed in Sec. 5.3 also motivated a different approach to the matter trapping problem, which is somewhat in between the purely dynamical solitonic binding mechanisms and the models where matter degrees are *a priori* pure brane excitations: If there are both brane and bulk contributions from the matter degrees of freedom to the action, then the physics of the matter degrees of freedom can look four-dimensional for a certain range of parameters, see [48] and [45] for recent discussions of this possibility.

The emergence of a viable four-dimensional gravitational potential in the Newtonian limit is the primary concern in any theory of extra dimensions with matter degrees of freedom restricted to a four-dimensional submanifold. The problem and its solutions are reviewed in Sec. 5, while Sec. 7 describes the cosmological implications of a four-dimensional world in a higher-dimensional space probed by gravitons.

Models like (2) with $\mathcal{M}_{3,1}$ representing the $(3+1)$-dimensional spacetime supporting matter degrees of freedom are now widely denoted as brane worlds, and as already indicated in (2) the present review will focus on thin 3-branes immersed in a $(4+1)$-dimensional spacetime, i.e. on models where matter degrees of freedom are strictly confined to a codimension 1 submanifold $\mathcal{M}_{3,1}$, or where the energies are so low compared to a dynamical binding mechanism that transverse matter excitations can be neglected.

2. Conventions

Our primary concern will be the discussion of dynamics in a $(4+1)$-dimensional spacetime $\mathcal{M}_{4,1}$ with matter restricted to a $(3+1)$-dimensional submanifold $\mathcal{M}_{3,1}$. However, most equations and results will be expressed for codimension 1 hypersurfaces $\mathcal{M}_{d-1,1}$ immersed in a spacetime $\mathcal{M}_{d,1}$.

Conventions for the metric and curvature are those of Misner, Thorne, Wheeler [85], i.e. the metric has signature $(-+\ldots+)$, the connection with Christoffel symbols $\Gamma^K{}_{LM}$ is always metric, and the Riemann and Ricci tensors are

$$R^K{}_{LMN} = \partial_M \Gamma^K{}_{LN} - \partial_N \Gamma^K{}_{LM} + \Gamma^K{}_{SM}\Gamma^S{}_{LN} - \Gamma^K{}_{SN}\Gamma^S{}_{LM}$$

and

$$R_{MN} = R^K{}_{MKN},$$

respectively.

The geodesic or proper distance from the codimension 1 submanifold $\mathcal{M}_{d-1,1}$ will be denoted as $d^\perp$, and we will define $x^\perp = d^\perp$ on one side of $\mathcal{M}_{d-1,1}$, and $x^\perp = -d^\perp$ on the other side. In any coordinate patch comprising a patch of $\mathcal{M}_{d-1,1}$, $x^d \equiv x^\perp$ will be used as the dth coordinate, whereas the first $d-1$ coordinates x^μ cover patches of constant proper distance¶ from $\mathcal{M}_{d-1,1}$. This yields Gaussian normal coordinates in a neighborhood of $\mathcal{M}_{d-1,1}$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + (dx^\perp)^2, \tag{3}$$

and *vice versa*: Due to

$$\Gamma^\perp{}_{\perp\perp} = 0, \;\; \Gamma^\mu{}_{\perp\perp} = 0$$

$|x^\perp|$ in (3) is a geodesic distance along orthogonal trajectories to $\mathcal{M}_{d-1,1}$.

The extrinsic curvature tensor in the metric (3) is

$$K_{\mu\nu} = -\frac{1}{2}\partial_\perp g_{\mu\nu}. \tag{4}$$

My convention for the d-dimensional Planck mass is such that the Einstein–Hilbert action is

$$S_{EH} = \frac{m_d^{d-1}}{2}\int d^{d+1}x\,\sqrt{-g}R.$$

¶ A particular way to construct such a Gaussian normal coordinate system is to first cover $\mathcal{M}_{d-1,1}$ with coordinate patches $\{x^\mu, 0 \le \mu \le d-1\}$ and then elevate these patches to a neighborhood of $\mathcal{M}_{d-1,1}$ along the perpendicular geodesics, with $x^\perp$ as the dth coordinate. $g_{\perp\perp} = 1$ and $g_{\perp\mu} = 0$ follow from the definition of distance and the geodesic equation.

Usually this implies a d-dimensional Newton constant

$$G_{N,d} = \frac{1}{2(d-1)\sqrt{\pi}^d m_d^{d-1}} \Gamma\left(\frac{d}{2}\right),$$

i.e. $m_3 = (8\pi G_{N,3})^{-1/2} = 2.4 \times 10^{18}$ GeV is the *reduced* Planck mass in $3+1$ dimensions (see, however, the model of Dvali, Gabadadze and Porrati (Sec. 5.3) for an exception).

With the exception of eq. (1) natural units $\hbar = c = 1$ are used.

3. The Lanczos–Israel matching conditions

The unique covariant second order equation for the metric in presence of covariantly conserved sources $\hat{T}_{MN}$ is the Einstein equation with a possible cosmological term:

$$R_{MN} - \frac{1}{2} g_{MN} \left(R - \frac{2\Lambda}{m_d^{d-1}}\right) = \frac{1}{m_d^{d-1}} \hat{T}_{MN}. \tag{5}$$

Spacetimes with sources confined to codimension 1 submanifolds are no exception to this rule. However, with matter restricted to a codimension 1 hypersurface

$$\hat{T}_{MN} = g_M{}^\mu g_N{}^\nu T_{\mu\nu} \delta(x^\perp)$$

eq. (5) yields Einstein spaces in the bulk:

$$R_{MN} = \frac{2\Lambda}{(d-1) m_d^{d-1}} g_{MN} \tag{6}$$

and a higher-dimensional version of the Lanczos–Israel matching conditions:

$$\lim_{\epsilon \to +0} \left[K_{\mu\nu}\right]_{x^\perp = -\epsilon}^{x^\perp = \epsilon} = \frac{1}{m_d^{d-1}} \left(T_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta} \right)\Bigg|_{x^\perp = 0}. \tag{7}$$

Here d refers to the number of spatial dimensions of the embedding space, i.e. $d = 4$ is the case of primary interest to us.

These matching conditions have been derived by Lanczos for the case of singular energy-momentum shells in general relativity ($d = 3$) [77, 78, 35], and a covariant derivation and the geometric formulation in terms of discontinuity of extrinsic curvature along the singular energy-momentum shell[+] were given by Israel [70].

Eq. (7) implies that the geometries in the two regions adjacent to an energy-momentum carrying codimension 1 hypersurface differ in such a way that the extrinsic curvature of that hypersurface is different on both sides. Expressed in more popular terms: What locally might be spherical from one side might be flat from the other side. In that sense an energy-momentum carrying codimension 1 hypersurface $\mathcal{M}_{d-1,1}$ could just as well be considered as a boundary between two adjacent spacetimes $\mathcal{M}_{d,1}^+$ and $\mathcal{M}_{d,1}^-$. $\mathcal{M}_{d,1}^+$ and $\mathcal{M}_{d,1}^-$ are continuously connected along $\mathcal{M}_{d-1,1}$, and they are smoothly connected only in those regions of $\mathcal{M}_{d-1,1}$ where no energy-momentum currents are present.

Using the equations of Gauss and Codazzi, and a result of Sachs for $R^\perp{}_{\mu\perp\nu}$ in Gaussian normal coordinates ([102], cf. [85]), we can express the bulk equations (6)

[+] According to [85] some of this was also anticipated by G. Darmois.

in a neighborhood of $\mathcal{M}_{d-1,1}$ in terms of intrinsic and extrinsic curvatures of the hypersurfaces $x^{\perp} =$ const. on either side of $\mathcal{M}_{d-1,1}$:

$$R_{\mu\nu} = R^{(d-1)}_{\mu\nu} + \partial_{\perp} K_{\mu\nu} + 2K_{\lambda\mu}K^{\lambda}{}_{\nu} - KK_{\mu\nu} = \frac{2\Lambda}{(d-1)m_d^{d-1}} g_{\mu\nu}, \tag{8}$$

$$R_{\mu\perp} = \partial_{\mu} K - \nabla_{\nu} K^{\nu}{}_{\mu} = 0, \tag{9}$$

$$R_{\perp\perp} = g^{\mu\nu}\partial_{\perp} K_{\mu\nu} + K^{\mu\nu}K_{\mu\nu} = \frac{2\Lambda}{(d-1)m_d^{d-1}}. \tag{10}$$

In applications of these equations to the hypersurface $\mathcal{M}_{d-1,1}$ itself, the extrinsic curvature terms should be replaced by the mean extrinsic curvature at each point of $\mathcal{M}_{d-1,1}$:

$$\overline{K}_{\mu\nu} = \frac{1}{2}\lim_{\epsilon\to+0}\left[K_{\mu\nu}|_{x^{\perp}=-\epsilon} + K_{\mu\nu}|_{x^{\perp}=\epsilon}\right]. \tag{11}$$

The Gauss equation (8) can be used to derive an effective relation between the intrinsic Einstein tensor on the brane, the local energy-momentum tensor, and the extrinsic curvatures on the brane [103, 13].

If our codimension 1 hypersurface $\mathcal{M}_{d-1,1}$ represents an energy-momentum carrying boundary with no adjacent region of spacetime on the other side we may simply delete the corresponding extrinsic curvature term, and (7) represents a boundary condition on the normal derivative of the metric.

4. The action principle with codimension 1 hypersurfaces: Need for the Gibbons–Hawking term

Since the bulk Einstein equation (5) follows from a bulk Einstein–Hilbert action, the natural expectation was that (5) with *a priori* codimension 1 sources (or equivalently (6) and (7)) could be directly derived from stationarity of

$$S_{EH} = \int dt \int d^{d-1}\mathbf{x} \int dx^{\perp}\sqrt{-g}\left(\frac{m_d^{d-1}}{2}R - \Lambda\right) \tag{12}$$
$$+ \int dt \int d^{d-1}\mathbf{x}\,\mathcal{L}\Big|_{x^{\perp}=0},$$

with the brane Lagrangian $\mathcal{L}$ containing only matter degrees of freedom and eventually intrinsic curvature terms of the brane. However, the fact that (5) follows from a bulk Einstein–Hilbert action *without* distinguished submanifold does not imply that it would also follow from (12) *with* the distinguished hypersurface $\mathcal{M}_{d-1,1}$: Every action that differs from the Einstein–Hilbert action by a complete divergence would yield (5), but once we designate the hypersurface $\mathcal{M}_{d-1,1}$ *a priori* in our action principle, the difference in surface terms between different bulk actions becomes relevant, because the presence of energy-momentum on $\mathcal{M}_{d-1,1}$ may spoil the continuity of the surface terms across $\mathcal{M}_{d-1,1}$, thus implying a numerical difference between the different bulk actions. Therefore not every bulk action which yields (5) without *a priori* designation of a hypersurface can yield (6) and (7) from a corresponding action principle of the sort (2), and this applies in particular to the Einstein–Hilbert term: Careful evaluation of the variation of S_{EH} yields [43, 42]

$$\begin{aligned}
\delta S_{EH} = \frac{m_d^{d-1}}{2}\lim_{\epsilon\to+0}\Bigg(\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\le-\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\\
\times\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\\
+\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\ge\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\Bigg)\\
+\frac{m_d^{d-1}}{2}\lim_{\epsilon\to+0}\int dt\int d^{d-1}\mathbf{x}\Big[\sqrt{-g}\left(g^{MN}\delta\Gamma^\perp{}_{MN}-g^{\perp N}\delta\Gamma^M{}_{MN}\right)\Big]_{x^\perp=\epsilon}^{x^\perp=-\epsilon}\\
+\int dt\int d^{d-1}\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^\perp=0}\\
=\frac{m_d^{d-1}}{2}\lim_{\epsilon\to+0}\Bigg(\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\le-\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\\
\times\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\\
+\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\ge\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\Bigg)\\
+\frac{m_d^{d-1}}{4}\lim_{\epsilon\to+0}\int dt\int d^{d-1}\mathbf{x}\Big[\sqrt{-g}(3\delta g^{\mu\nu}\partial_\perp g_{\mu\nu}-\delta g^{\perp\perp}g^{\mu\nu}\partial_\perp g_{\mu\nu}\\
+2g_{\mu\nu}\partial_\perp\delta g^{\mu\nu})\Big]_{x^\perp=\epsilon}^{x^\perp=-\epsilon}+\int dt\int d^{d-1}\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^\perp=0}.
\end{aligned}\tag{13}$$

The junction conditions following from $\delta S_{EH}=0$ are incompatible with the junction condition (7). Even if we neglect the $\delta g^{\perp\perp}$ junction term, the $\delta g^{\mu\nu}$ junction term appears with the wrong coefficient and a missing trace term for (7), and there appears a term proportional to $\partial_\perp\delta g^{\mu\nu}$ which usually has no match in $\delta\mathcal{L}/\delta g^{MN}$. The difficulty with the Einstein action with metric discontinuities was noticed in four dimensions in a Euclidean ADM formalism already by Hayward and Louko [66].

One could argue against eq. (13) that (5) implies a curvature singularity and hence a singularity of R on $\mathcal{M}_{d-1,1}$, whence additional boundary terms should be included in (12) and (13). However, if the matter on $\mathcal{M}_{d-1,1}$ is radiation dominated, then R has the same constant value

$$R=\frac{d+1}{d-1}\frac{2\Lambda}{m_d^{d-1}}$$

everywhere in $\mathcal{M}_{d,1}$, and no genuine δ-function contribution to (12) or (13) arises classically from the Einstein–Hilbert term. It also does not help to include an intrinsic curvature term $\int_{\mathcal{M}_{d-1,1}} d^d x\, R^{(d-1)}$ on the brane, since variation of the brane intrinsic curvature scalar cannot compensate for the $\partial_\perp \delta g^{\mu\nu}$ junction term from the bulk curvature scalar.

It was pointed out in [42] that replacing the Einstein–Hilbert term with an Einstein term

$$\mathcal{L}_E = \frac{m_d^{d-1}}{2}\sqrt{-g}g^{MN}\left(\Gamma^K{}_{LM}\Gamma^L{}_{KN} - \Gamma^K{}_{KL}\Gamma^L{}_{MN}\right) \tag{14}$$

$$= \mathcal{L}_{EH} - \frac{m_d^{d-1}}{2}\partial_L\left(\sqrt{-g}g^{MN}\Gamma^L{}_{MN} - \sqrt{-g}g^{LN}\Gamma^M{}_{MN}\right),$$

in the bulk directly yields the covariant equations (6,7).

However, a more appealing solution to the problem to derive (6,7) directly from an action principle of the kind (2) with an *a priori* designated hypersurface $\mathcal{M}_{d-1,1}$ employs a Gibbons–Hawking term $\sim \int_{\mathcal{M}_{d-1,1}} d^d x\, K$ [56, 21], see also [66, 59, 105]. In doing so we should take the mean extrinsic curvature $\overline{K}$ from (11), because

$$\lim_{\epsilon\to+0}\,[K]_{x^\perp=-\epsilon}^{x^\perp=\epsilon} = -\frac{1}{(d-1)m_d^{d-1}}g^{\mu\nu}T_{\mu\nu}\Big|_{x^\perp=0}. \tag{15}$$

With the proper normalization the appropriate Gibbons–Hawking term is

$$S_{GH} = -m_d^{d-1}\int dt\int d^{d-1}\mathbf{x}\,\sqrt{-g}\,\overline{K} \tag{16}$$

and variation of the metric yields

$$\begin{aligned}\delta S_{EH} + \delta S_{GH} = \frac{m_d^{d-1}}{2}\lim_{\epsilon\to+0}&\left(\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\le-\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\right.\\
&\times\left(R_{MN} - \frac{1}{2}g_{MN}R + \frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\\
&\left.+\int dt\int d^{d-1}\mathbf{x}\int_{x^\perp\ge\epsilon}dx^\perp\sqrt{-g}\delta g^{MN}\left(R_{MN} - \frac{1}{2}g_{MN}R + \frac{\Lambda}{m_d^{d-1}}g_{MN}\right)\right)\\
&+\frac{m_d^{d-1}}{4}\lim_{\epsilon\to+0}\int dt\int d^{d-1}\mathbf{x}\Big[\sqrt{-g}\delta g^{\mu\nu}\left(\partial_\perp g_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\partial_\perp g_{\alpha\beta}\right)\Big]_{x^\perp=\epsilon}^{x^\perp=-\epsilon}\\
&+\int dt\int d^{d-1}\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\Big|_{x^\perp=0}.\end{aligned} \tag{17}$$

$\delta S_{EH} + \delta S_{GH} = 0$ yields exactly the Einstein condition (6) in the bulk and the matching condition (7) on the brane*.

* For immersed hypersurfaces $S = S_{EH} + S_{GH}$ yields exactly the same variation as the corresponding Einstein action S_E, i.e. the action without a Gibbons–Hawking term and with an Einstein term (14) in the bulk. However, when the hypersurface is a true boundary of $\mathcal{M}_{d,1}$ a further boundary term $\sim \sqrt{-g}\delta g^{\perp\mu}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$ appears in δS_E, which would require constant d-volume of the boundary.

5. The Newtonian limit on thin branes

The background geometry of a spacetime satisfying (6,7) is approximately flat on length scales $r \ll \left(m_d^{d-1}/|\Lambda|\right)^{1/2}$, while on the other hand our classical calculations certainly become meaningless at scales $\approx m_d^{-1}$. Hence the conditions for an ordinary flat Newtonian limit are♯

$$\frac{1}{m_d} \ll r \ll \left(\frac{m_d^{d-1}}{|\Lambda|}\right)^{\frac{1}{2}}. \tag{18}$$

On those length scales where the background geometry is approximately flat, the gravitational potential $U = -h_{00}/2$ of a mass distribution $\varrho(\mathbf{r})$ in $d \geq 3$ spatial dimensions is *usually* given by the d-dimensional elliptic Green's function for Dirichlet boundary conditions at infinity:

$$G(\mathbf{r}) = \frac{1}{4\sqrt{\pi}^d}\Gamma\left(\frac{d-2}{2}\right)\frac{1}{r^{d-2}} \tag{19}$$

through

$$U(\mathbf{r}) = -\frac{1}{2(d-1)\sqrt{\pi}^d m_d^{d-1}}\Gamma\left(\frac{d}{2}\right)\int d^d\mathbf{r}'\,\frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{d-2}}. \tag{20}$$

$U(\mathbf{r})$ arises from the 00-component of the $(d+1)$-dimensional Einstein equation in its linearized static form:

$$\Delta U(\mathbf{r}) = \frac{1}{m_d^{d-1}}\frac{d-2}{d-1}\varrho(\mathbf{r}), \tag{21}$$

and the corresponding potential energy of a mass m is $mU(\mathbf{r})$.

Eqs. (19,20) tell us that interactions in higher-dimensional spacetimes are usually weaker at larger distances and stronger at shorter distances, and Kepler's laws would not hold. *A priori* this sustains in our brane models if we do not invoke special mechanisms or geometrical constraints to ensure an r^{3-d}-limit for the Newton potential on the hypersurface $\mathcal{M}_{d-1,1}$:

In models with energy-momentum bound to a hypersurface $\mathcal{M}_{d-1,1}$ the Newtonian limit arises from the static weak field approximation to (6,7).
This yields in the bulk:

$$(\Delta + \partial_\perp^2)U(\mathbf{r}, x^\perp) = 0, \tag{22}$$

and along the junction $\mathcal{M}_{d-1,1}$:

$$\lim_{\epsilon\to+0}\left[\partial_\perp U(\mathbf{r}, x^\perp)\right]_{x^\perp=-\epsilon}^{x^\perp=\epsilon} = \frac{1}{m_d^{d-1}}\frac{d-2}{d-1}\varrho(\mathbf{r}), \tag{23}$$

♯ E.g. in our spacetime with an eventual positive cosmological constant $\Lambda < 10^{-120}m_3^4$ these conditions are comfortably fulfilled on all scales where the Newtonian limit is tested and supposed to hold [112, 68], the upper limit being $r \ll 10^9$ light years. In this case and in our epoch the upper limit from the bulk background matter is the same as the limit from the maximally allowed cosmological constant.

or equivalently

$$(\Delta+\partial_\perp^2)U(\mathbf{r},x^\perp)=\frac{1}{m_d^{d-1}}\frac{d-2}{d-1}\varrho(\mathbf{r})\delta(x^\perp). \tag{24}$$

This, of course, yields nothing but (20) with the split†† $\mathbf{r}\to\mathbf{r}+x^\perp\mathbf{e}_\perp$ and codimension 1 sources:

$$U(\mathbf{r},x^\perp)=\frac{-1}{2(d-1)\sqrt{\pi}^d m_d^{d-1}}\Gamma\left(\frac{d}{2}\right)\int d^{d-1}\mathbf{r}'\,\frac{\varrho(\mathbf{r}')}{[(\mathbf{r}-\mathbf{r}')^2+{x^\perp}^2]^{(d-2)/2}}, \tag{25}$$

and the gravitational potential within $\mathcal{M}_{d-1,1}$ would inherit the higher-dimensional distance law:

$$U(\mathbf{r})=-\frac{1}{2(d-1)\sqrt{\pi}^d m_d^{d-1}}\Gamma\left(\frac{d}{2}\right)\int d^{d-1}\mathbf{r}'\,\frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{d-2}}. \tag{26}$$

However, mechanisms have been proposed in recent years to generate a correct $(d-1)$-dimensional Newtonian limit on $\mathcal{M}_{d-1,1}$ even for length scales $\ell^\perp$ of extra dimensions much larger than the low-dimensional Planck length $\ell_{d-1}=1/m_{d-1}$ or the length scales $\hbar c/E$ which ordinarily should be ruled out through accelerators:

5.1. *The observation of Arkani-Hamed, Dimopoulos and Dvali*

If the extra dimension has a finite extension $\ell^\perp$ which is well below the minimal currently accessible length scale $\approx 0.2\,\mathrm{mm}$ for tests of Einstein gravity [68], then the Dirichlet Green's function (19) with vanishing boundary condition for $x^\perp\to\infty$ is certainly not appropriate, and we should expect that the low-dimensional Newtonian potential at scales $r>\ell^\perp$ should result from a gravitational field which is quenched over the transverse dimension [10, 6, 11]. This should yield the expected $(d-1)$-dimensional gravitational potential on $\mathcal{M}_{d-1,1}$. I will denote this as an ADD type mechanism. In that case even the fundamental quantum gravity scale m_4 of the theory can be much smaller than our 4-dimensional Planck mass m_3 [10, 11].

Following [11] we can derive the relation between the Newton constants in $d-1$ and d spatial dimensions under the assumption that the extra dimension has finite length $\ell^\perp$ by calculating the flux of the gravitational field of a mass M through a d-dimensional cylinder of radius r and length $\ell^\perp$:

From (21) we get for $r>\ell^\perp$

$$\frac{2\sqrt{\pi}^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)}\ell^\perp r^{d-2}\frac{dU}{dr}=\frac{d-2}{d-1}\frac{M}{m_d^{d-1}},$$

i.e.

$$U(r)=-\frac{d-2}{2(d-1)(d-3)\sqrt{\pi}^{d-1}m_d^{d-1}\ell^\perp}\Gamma\left(\frac{d-1}{2}\right)\frac{M}{r^{d-3}}.$$

†† Things become a little more subtle if $\mathcal{M}_{d-1,1}$ is a compact boundary of a spacetime with a non-periodic extra dimension. The emergence of a r^{-1}-limit from a Neumann–type Green's function in such a setting is discussed in [43].

Comparison with (20) for $d-1$ spatial dimensions yields

$$(d-2)^2 m_{d-1}^{d-2} = (d-1)(d-3) m_d^{d-1} \ell^{\perp}. \tag{27}$$

For $d = 4$ and $\ell^{\perp} < 0.2\,\mathrm{mm}$ this yields a lower bound on a 5-dimensional Planck scale which is well below m_3:

$$m_4 = \left(\frac{4m_3^2}{3\ell^{\perp}}\right)^{\frac{1}{3}} > 2 \times 10^8\,\mathrm{GeV}. \tag{28}$$

The result for general number ν of extra spacelike dimensions of length $\ell^{\perp}$ is

$$m_{3+\nu} = \left(\frac{2(\nu+1)m_3^2}{(\nu+2){\ell^{\perp}}^{\nu}}\right)^{\frac{1}{\nu+2}} > 10^{(37-12\nu)/(\nu+2)}\,\mathrm{GeV}.$$

Two concentric 3-spheres of radii $a < b$ provide a model system where the realization of the ADD mechanism for $a, b \gg b - a$ and the realization of the higher-dimensional singularity for $b \gg a$ can be studied analytically. This model realizes the 3-brane as a boundary of a spatial four-dimensional bulk, and therefore the potential arises from the Green's function for Neumann boundary conditions, but it can be written down exactly in terms of the four-dimensional multipole expansion with azimuthal symmetry [43].

5.2. The Randall–Sundrum model

Another possibility arises if our low-dimensional Newtonian limit is not flat, because a bulk cosmological term induces a transverse length scale smaller than the length scales tested in experimental gravity. Again the flat background approximation in our calculation of the Newtonian limit would be invalidated, and the classical approximation might be invalidated as well. In consideration of [98, 99] I will denote this as an RS type mechanism. Earlier discussions of the emergence of metrics of the form $ds^2 = \phi(x^{\perp})\eta_{\mu\nu}dx^{\mu}dx^{\nu} + {dx^{\perp}}^2$ in 5-dimensional models can be found in [101, 60]. $\phi(x^{\perp})$ is usually denoted as a warp factor in these models.

Randall and Sundrum have proposed a 3-brane with a brane tension λ_3 in a 5-dimensional bulk with a cosmological constant Λ. The metric

$$ds^2 = \exp\left(-\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right)\eta_{\mu\nu}dx^{\mu}dx^{\nu} + {dx^{\perp}}^2 \tag{29}$$

solves (5) for $d = 4$ with a bulk cosmological constant

$$\Lambda = -\frac{\lambda_3^2}{6m_4^3} \tag{30}$$

and

$$\hat{T}_{MN} = -g_M{}^{\mu} g_N{}^{\nu} \eta_{\mu\nu} \lambda_3 \delta(x^{\perp}).$$

The original setup of Randall and Sundrum consisted of a $\mathbb{Z}_2$-symmetric configuration of two 3-branes embedded in $\mathcal{M}_{3,1} \times S_1$. If the transverse extension $2\pi R_{\circ}$ is very small compared to $0.2\,\mathrm{mm}$, then the argument of ADD applies to ensure an ordinary Newtonian limit at distances $r \gg R_{\circ}$. However, it was argued in [99] that a 3-brane in an infinitely extended bulk (29) yields a viable approximation to ordinary Newtonian gravity in four dimensions through a trapped massless graviton mode (see also [73, 31, 12, 62, 83, 34, 79, 29] for corresponding setups of more branes and the stabilization problem in theories with several branes). This proposal implies that curvature should play an important role in the low-energy limit, and this has to be taken into account in the discussion of graviton evolution equations in this type of models. The problem of the weak field expansion around the corresponding curved background has been studied by many groups [103, 53, 58, 64, 87, 8, 62, 30, 63, 28, 72, 23, 55, 91, 92].

Therefore the primary problem to address in the present setting is: Can the transverse distance $|x^{\perp}|$ from a flat 3-brane in the metric (29) exceed 0.2 mm without contradicting experimental tests of Einstein gravity, due to the curvature of the background geometry?

Motion along the brane in the stationary weak field expansion around the background geometry (29) can be described by a gravitational potential‡ (with $h_{MN} = \delta g_{MN}$)

$$U = -\frac{1}{2}\exp\left(\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right) h_{00}.$$

However, as indicated above, a peculiarity arises in the discussion of the phenomenological suitability of the Randall–Sundrum model with a large extra dimension: Curvature must play an essential role if the model as proposed with a large extra dimension is to reproduce a four-dimensional gravitational potential at those length scales where we observe the potential. This implies that in the calculation of U we have to expand our fields around the curved background (29) rather than around an approximately flat section of that background. However, gravitons are tensor particles which do not satisfy decoupled evolution equations in a curved background, and it is also not possible to derive a decoupled equation for the gravitational potential. This is obvious also from the analogy between the problem to derive a graviton evolution equation in a curved background and the theory of cosmological perturbations, and another direct way to see non-separability in a curved background is to recall the formula

$$\delta R_{MN} = \nabla_K \delta\Gamma^K{}_{MN} - \nabla_N \delta\Gamma^K{}_{KM}$$

for the first order variation of the Ricci tensor under first order changes of the metric§

‡ The gravitational potential cannot fully account for motion in the transverse direction, if particles could leave the brane.

§ Even in a flat background Gaussian normal coordinates would not be the best choice when it comes to separation of the Einstein equation in the weak field approximation. However, as emphasized above, the problem at hand also has a gauge independent origin in the background curvature. A harmonic gauge for the longitudinal coordinates x^{μ} (with $h^{\alpha}{}_{\mu} = \eta^{\alpha\beta}h_{\beta\mu}$, etc.):

$$\partial_\alpha h^{\alpha}{}_{\mu} - \frac{1}{2}\partial_\mu h^{\alpha}{}_{\alpha} = \frac{1}{2}\exp\left(-\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right)\partial_\mu h_{\perp\perp} - \partial_\perp\left[\exp\left(-\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right) h_{\perp\mu}\right]$$

is useful in separating the 2nd order derivative terms, but the evolution of $h_{\mu\nu}$ will not decouple from $h_{\perp\perp}$, $h_{\perp\mu}$ and $h^{\alpha}{}_{\alpha}$. In principle, one could eliminate the latter by solving the coupled set of equations involving $R_{\perp\perp}$, $R_{\perp\mu}$ and $R^{\alpha}{}_{\alpha}$, but that means trading the couplings for non-local source terms.

To get a qualitative understanding of the expected behavior of the gravitational potential, we may consider the following equation for the gravitational potential, which arises from the diagonal terms of R_{00} for time-independent h_{00}:

$$\Delta U(\mathbf{r},x^{\perp})+\exp\left(\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right)\partial_{\perp}\left[\exp\left(-\frac{2\lambda_3}{3m_4^3}|x^{\perp}|\right)\partial_{\perp}U(\mathbf{r},x^{\perp})\right] \tag{31}$$
$$=\frac{2}{3m_4^3}\varrho(\mathbf{r})\delta(x^{\perp}).$$

Note that the differential operator on the left hand side is just $\exp(-\lambda_3|x^{\perp}|/(3m_4^3))$ times the scalar covariant Laplacian in the metric (29), applied to a time-independent field.

Inspecting this equation for an extended, weakly $\mathbf{r}$-dependent source shows that the potential orthogonal to the source evolves exponentially

$$U(x^{\perp})\sim\frac{1}{\lambda_3}\exp\left(\frac{2\lambda_3}{3m_4^3}|x^{\perp}|\right).$$

This indicates that the resulting gravitational potential of a mass distribution $\varrho(\mathbf{r})$ on the negative tension brane should remain localized within a penetration depth

$$\ell_{RS}^{\perp}=\frac{3m_4^3}{-2\lambda_3}, \tag{32}$$

which should be smaller than 0.2 mm. In a reasoning similar to the ADD argument we would expect an effective 4-dimensional Planck mass

$$m_3\approx\sqrt{m_4^3\ell_{RS}^{\perp}},$$

implying $m_4>10^8\,\mathrm{GeV}$, cf. (28), and

$$\lambda_3\approx-\frac{m_3^2}{{\ell_{RS}^{\perp}}^2}<-10^{13}\,\mathrm{GeV}^4, \tag{33}$$

corresponding to a bulk cosmological constant

$$\Lambda\approx-\frac{m_3^2}{{\ell_{RS}^{\perp}}^3}<-10\,\mathrm{GeV}^5. \tag{34}$$

It was noticed by Mück *et al.* that a fully fledged linearized theory indicates that the brane tension should also be negative in the single brane setup [87, 8]. The negative tension brane is also distinguished by the fact that timelike geodesics are pulled towards the brane [87, 63]:

$$\frac{d^2x^{\perp}}{ds^2}=-\frac{\lambda_3}{6m_4^3}\eta_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\mathrm{sign}(x^{\perp})\exp\left(-\frac{\lambda_3}{3m_4^3}|x^{\perp}|\right)$$
$$\Rightarrow\ \mathrm{sign}(\ddot{x}^{\perp})=\mathrm{sign}(\lambda_3)\mathrm{sign}(x^{\perp}).$$

If it were for ordinary Einstein–Friedmann cosmology, a (3+1)-dimensional universe with a negative cosmological constant satisfying (33) would have collapsed long ago. Here, however, the effect of λ_3 is to balance the effect from the bulk cosmological constant

Λ to keep the hypersurface $\mathcal{M}_{3,1}$ flat in the zeroth order approximation. The fact that $|\lambda_3| \approx |\Lambda|\ell_{RS}^{\perp}$ exceeds effective four-dimensional mean energy densities‖

$$\varrho_4 \simeq 81h^2\,\mathrm{meV}^4,$$

by at least a factor 10^{59} shows that matter densities can indeed be treated as perturbations in cosmological investigations of this scenario.

5.3. The model of Dvali, Gabadadze and Porrati

The idea behind the model of Dvali, Gabadadze and Porrati is competition between the bulk curvature scalar R and the corresponding intrinsic curvature scalar $R^{(d-1)}$ on the brane [47, 46]. We will again focus on the codimension 1 case and calculate the gravitational potential.

In the light of the results of Sec. 4 we should write the action of the model as

$$S = \frac{m_d^{d-1}}{2}\int dt \int d^{d-1}\mathbf{x} \int dx^{\perp}\sqrt{-g}R \tag{35}$$
$$+ \int dt \int d^{d-1}\mathbf{x}\left(\frac{m_{d-1}^{d-2}}{2}\sqrt{-g}R^{(d-1)} - m_d^{d-1}\sqrt{-g}\,\overline{K} + \mathcal{L}\right)\Bigg|_{x^{\perp}=0},$$

with the Lagrangian $\mathcal{L}$ containing the matter degrees of freedom. The model was motivated by radiative generation of a kinetic graviton term on the brane [47, 18, 1, 2].

The action (35) yields Einstein equations

$$m_d^{d-1}\left(R_{MN} - \frac{1}{2}g_{MN}R\right) + m_{d-1}^{d-2}g_M{}^{\mu}g_N{}^{\nu}\left(R_{\mu\nu}^{(d-1)} - \frac{1}{2}g_{\mu\nu}R^{(d-1)}\right)\delta(x^{\perp})$$
$$= g_M{}^{\mu}g_N{}^{\nu}T_{\mu\nu}\delta(x^{\perp}), \tag{36}$$

and the resulting matching condition (7) is

$$\lim_{\epsilon\to+0}\,[K_{\mu\nu}]_{x^{\perp}=-\epsilon}^{x^{\perp}=\epsilon} = \frac{1}{m_d^{d-1}}\left(T_{\mu\nu} - \frac{1}{d-1}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}\right)\Bigg|_{x^{\perp}=0} \tag{37}$$
$$- \frac{m_{d-1}^{d-2}}{m_d^{d-1}}\left(R_{\mu\nu}^{(d-1)} - \frac{1}{2(d-1)}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}^{(d-1)}\right)\Bigg|_{x^{\perp}=0}.$$

It is amusing that the model of Dvali, Gabadadze and Porrati provides a novel and entirely unprecedented realization of the old proposal of Lorentz and Levi-Civita to consider $-m_3^2[R_{\mu\nu}^{(3)} - g_{\mu\nu}(R^{(3)}/2)]$ as the energy-momentum tensor of the gravitational field.

For the weak field approximation I still prefer to employ Gaussian normal coordinates for the background metric, because of the inevitable factor $\delta(x^{\perp})$ in the

‖ The parameter $0.6 \leq h \leq 0.8$ parametrizes the uncertainty in the value of the Hubble constant $H = 100h\,\mathrm{km/(s\,Mpc)}$.

Einstein equation. This implies that we can impose a harmonic gauge condition only on the longitudinal coordinates x^μ:

$$\partial_\alpha h^\alpha{}_\mu + \partial_\perp h_{\perp\mu} = \frac{1}{2}\partial_\mu \left(h^\alpha{}_\alpha + h_{\perp\perp}\right), \tag{38}$$

but this is sufficient to get a decoupled equation for the gravitational potential of a static mass distribution:

The transverse equations in the gauge (38)

$$R_{\perp\perp} - R^\alpha{}_\alpha = \frac{1}{2}\partial_\alpha\partial^\alpha \left(h^\beta{}_\beta - h_{\perp\perp}\right) + \partial_\perp\partial_\alpha h^\alpha{}_\perp = 0,$$

$$R_{\perp\mu} = \frac{1}{2}\left(\partial_\mu\partial_\alpha h^\alpha{}_\perp - \partial_K\partial^K h_{\perp\mu}\right) + \frac{1}{4}\partial_\mu\partial_\perp \left(h_{\perp\perp} - h^\alpha{}_\alpha\right) = 0$$

can be solved by $h_{\perp\mu} = 0$, $h_{\perp\perp} = h^\alpha{}_\alpha$, whence the remaining equations take the form

$$m_d^{d-1}(\partial_\alpha\partial^\alpha + \partial_\perp^2)h_{\mu\nu} + m_{d-1}^{d-2}\delta(x^\perp)\left(\partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^\alpha{}_\alpha\right)$$

$$= -2\delta(x^\perp)\left(T_{\mu\nu} - \frac{1}{d-1}\eta_{\mu\nu}\eta^{\alpha\beta}T_{\alpha\beta}\right).$$

For $d = 4$ this yields the equation for the gravitational potential of a mass density $\varrho(\mathbf{r}) = M\delta(\mathbf{r})$ on $\mathcal{M}_{3,1}$:

$$m_4^3(\Delta + \partial_\perp^2)U(\mathbf{r}, x^\perp) + m_3^2\delta(x^\perp)\Delta U(\mathbf{r}, x^\perp) = \frac{2}{3}M\delta(\mathbf{r})\delta(x^\perp). \tag{39}$$

Insertion of a Fourier *ansatz*

$$U(\mathbf{r}, x^\perp) = \frac{1}{(2\pi)^4}\int d^3\mathbf{p}\int dp_\perp\, U(\mathbf{p}, p_\perp)\exp\left(i(\mathbf{p}\cdot\mathbf{r} + p_\perp x^\perp)\right)$$

yields an integral equation

$$m_4^3(\mathbf{p}^2 + p_\perp^2)U(\mathbf{p}, p_\perp) + \frac{m_3^2}{2\pi}\mathbf{p}^2\int dp'_\perp\, U(\mathbf{p}, p'_\perp) = -\frac{2}{3}M. \tag{40}$$

This equation tells us that $U(\mathbf{p}, p_\perp)$ must be of the form

$$U(\mathbf{p}, p_\perp) = \frac{f(\mathbf{p})}{\mathbf{p}^2 + p_\perp^2},$$

and $f(\mathbf{p})$ is then easily determined algebraically:

$$U(\mathbf{p}, p_\perp) = -\frac{4}{3}\frac{M}{(\mathbf{p}^2 + p_\perp^2)(2m_4^3 + m_3^2|\mathbf{p}|)}. \tag{41}$$

The resulting potential on the brane is

$$U(\mathbf{r}) = -\frac{M}{6\pi m_3^2 r}\left[\cos\left(\frac{2m_4^3}{m_3^2}r\right) - \frac{2}{\pi}\cos\left(\frac{2m_4^3}{m_3^2}r\right)\mathrm{Si}\left(\frac{2m_4^3}{m_3^2}r\right)\right. \tag{42}$$

$$\left. + \frac{2}{\pi}\sin\left(\frac{2m_4^3}{m_3^2}r\right)\mathrm{ci}\left(\frac{2m_4^3}{m_3^2}r\right)\right],$$

with the sine and cosine integrals

$$\mathrm{Si}(x) = \int_0^x d\xi\,\frac{\sin\xi}{\xi},$$

$$\mathrm{ci}(x) = -\int_x^\infty d\xi\,\frac{\cos\xi}{\xi}.$$

The model of Dvali, Gabadadze and Porrati predicts a transition scale

$$\ell_{DGP} = \frac{m_3^2}{2m_4^3} \tag{43}$$

between four-dimensional behavior and five-dimensional behavior of the gravitational potential:

$$\begin{aligned} r \ll \ell_{DGP}: \ U(\mathbf{r}) = {} & -\frac{M}{6\pi m_3^2 r}\left[1 + \left(\gamma - \frac{2}{\pi}\right)\frac{r}{\ell_{DGP}}\right. \\ & \left. + \frac{r}{\ell_{DGP}}\ln\!\left(\frac{r}{\ell_{DGP}}\right) + \mathcal{O}\!\left(\frac{r^2}{\ell_{DGP}^2}\right)\right], \\ r \gg \ell_{DGP}: \ U(\mathbf{r}) = {} & -\frac{M}{6\pi^2 m_4^3 r^2}\left[1 - 2\frac{\ell_{DGP}^2}{r^2} + \mathcal{O}\!\left(\frac{\ell_{DGP}^4}{r^4}\right)\right]. \end{aligned}$$

$\gamma \simeq 0.577$ is Euler's constant. If we would use the reduced Planck mass for m_3, then the small r potential would be stronger than the genuine four-dimensional potential by a factor $\frac{4}{3}$ because the coupling of the masses on the brane to the four-dimensional Ricci tensor is increased by this factor, cf. (21,39). This factor $\frac{4}{3}$ is in agreement with the tensorial structure of the graviton propagator reported in [47], which has been attributed to an additional helicity state of the five-dimensional graviton which in a first approximation appears like mediating an additional attractive scalar interaction from a four-dimensional perspective. While it might seem like a simple rescaling of the relation between m_3 and $G_{N,3}$, this additional state is clearly a matter of phenomenological concern [47]. Note, however, that a logarithmic modification of the Newton potential is usually not accounted for in the standard parametrized post-Newtonian formalism [112]. Furthermore, the logarithmic term does not resemble the type of modification that one expects from an effective four-dimensional scalar-tensor theory of gravity. Therefore the phenomenological implications of the model of Dvali, Gabadadze and Porrati with $m_3 = (6\pi G_{N,3})^{-1/2} \simeq 2.8 \times 10^{18}\,\mathrm{GeV}$ warrant further study.

6. A remark on black holes in the model of Dvali, Gabadadze and Porrati

Properties of sub-millimeter and primordial black holes in theories with sub-millimeter extra dimensions [10, 6, 11] were discussed by Argyres *et al.* [9]. Extensions of the Schwarzschild metric into the bulk of the Randall–Sundrum model (and variants of it) have been proposed and investigated in [20, 52, 39, 50, 54, 104, 22, 38, 17, 19, 49, 86, 96].

To my knowledge at the time of this writing no dedicated investigations of possible extensions of the Schwarzschild metric or black hole properties in the framework of the DGP model [47] have been reported. This may seem surprising given the attractiveness of this model. However, eqs. (41,42) show that even in the Schwarzschild case, which should translate into an axially symmetric metric in the DGP model, the result will be much more complicated than the four- or five-dimensional Schwarzschild metrics: If an analytic expression can be found at all, it inevitably will involve special functions. It is also clear that the restriction of the metric to the 3-brane will approximate the four-dimensional Schwarzschild metric at most for a certain range of r, where r is supposed to be the standard Schwarzschild radial coordinate on the brane.

The axial symmetry of (36) and spherical symmetry on the 3-brane imply that on every hypersurface $x^\perp = \text{const.}$ we should have a radial coordinate r (beyond an eventual event horizon) such that the sections $t = \text{const.}$, $r = \text{const.}$, $x^\perp = \text{const.}$ correspond to 2-spheres of circumference $2\pi r$ and area $4\pi r^2$. This entails a metric *ansatz*

$$ds^2 = -n^2(x^\perp, r)dt^2 + a^2(x^\perp, r)dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + dx^{\perp 2}. \qquad (44)$$

The coordinates employed in this *ansatz* are subject to the restrictions that $x^\perp$ is only applicable in that neighborhood of the brane which is covered by geodesics emerging from the brane, while r must have a lower limit in terms of an event horizon or the extension of the mass distribution generating the metric (44).

With the abbreviations for partial derivatives

$$\check{f}(x^\perp, r) = \frac{\partial}{\partial r} f(x^\perp, r),$$

$$f'(x^\perp, r) = \frac{\partial}{\partial x^\perp} f(x^\perp, r),$$

the non-vanishing components of the extrinsic curvature tensor of the hypersurfaces $x^\perp = \text{const.}$ are

$$K_{tt} = nn', \qquad (45)$$

$$K_{rr} = -aa', \qquad (46)$$

and we find for the non-vanishing components of the Ricci tensor

$$R_{\vartheta\vartheta} = 1 - \frac{1}{a^2} - \frac{\check{n}r}{na^2} + \frac{\check{a}r}{a^3}, \qquad (47)$$

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta}, \qquad (48)$$

$$R_{tt} = \frac{n\check{\check{n}}}{a^2} - \frac{n\check{n}\check{a}}{a^3} + 2\frac{n\check{n}}{a^2 r} + nn'' + \frac{n}{a}n'a' \qquad (49)$$

$$R_{rr} = -\frac{\check{\check{n}}}{n} + \frac{\check{n}\check{a}}{na} + 2\frac{\check{a}}{ar} - aa'' - \frac{a}{n}n'a', \qquad (50)$$

$$R_{r\perp} = -\frac{\check{n}'}{n} + \frac{a'}{a}\left(\frac{\check{n}}{n} + \frac{2}{r}\right), \qquad (51)$$

$$R_{\perp\perp} = -\frac{a''}{a} - \frac{n''}{n}. \qquad (52)$$

The equations (36,37) then translate into an equation which holds in the whole region of applicability of the coordinates used in (44):

$$\frac{a^2 - 1}{r} = \frac{\check{n}}{n} - \frac{\check{a}}{a}, \qquad (53)$$

equations which hold in the bulk:

$$\frac{a''}{a} = -\frac{n''}{n} = \frac{1}{a^2 r}\left(\frac{\check{a}}{a} + \frac{\check{n}}{n}\right), \tag{54}$$

$$\frac{\check{\check{n}}}{na^2} - \frac{\check{n}\check{a}}{na^3} + \frac{n'a'}{na} = \frac{1}{a^2 r}\left(\frac{\check{a}}{a} - \frac{\check{n}}{n}\right), \tag{55}$$

$$\frac{\check{n}'}{n} = \frac{a'}{a}\left(\frac{\check{n}}{n} + \frac{2}{r}\right), \tag{56}$$

and equations holding only on the brane:

$$\left[\frac{\check{\check{n}}}{n} - \frac{\check{n}\check{a}}{na} + \frac{a^2 - 1}{r^2}\right]_{x^\perp = 0} = 0, \tag{57}$$

$$\frac{a'}{a}(x^\perp \to \pm 0, r) = -\frac{n'}{n}(x^\perp \to \pm 0, r) = \pm \frac{m_3^2}{2m_4^3}\left[\frac{1}{a^2 r}\left(\frac{\check{a}}{a} + \frac{\check{n}}{n}\right)\right]_{x^\perp = 0}. \tag{58}$$

These equations allow for a black string solution which would plainly continue the four-dimensional Schwarzschild metric into the bulk along the orthogonal geodesics. However, this is an artefact of the fact that the coordinates in (44) have an event horizon r_M, and it is clearly not the correct solution for a brane black hole: It would give a four-dimensional Newtonian potential on each hypersurface $x^\perp = \mathrm{const.}$ in the large r limit, instead of fulfilling the correct boundary condition of a five-dimensional Newtonian potential at large distance.

For $r \ll \ell_{DGP}$ we notice that eqs. (53,57,58) approximate the ordinary equations for the Schwarzschild metric on the brane, i.e. the correct solution on the brane will approximate the four-dimensional Schwarzschild solution for $r_M < r \ll \ell_{DGP}$:

$$n^2(0, r) \approx 1 + 2U(r) + \mathcal{O}(r/\ell_{DGP}),$$

cf. (42,43).

For $r \gg \ell_{DGP}$ we notice that (58) implies that the metric becomes smooth across the brane in that limit, while (57) reduces to a special case of (53,55). The remaining equations are just the conditions for a Ricci flat five-dimensional spacetime, and the solution must approximate an axially symmetric five-dimensional black hole spacetime, in agreement with the role of ℓ_{DGP} as a scale separating four-dimensional effects from five-dimensional effects in the DGP model.

7. The cosmology of codimension 1 brane worlds

The five-dimensional Einstein tensor for the line element (with $x_i \equiv x^i$, $r^2 \equiv x_i x^i$)

$$\begin{aligned} ds^2 &= -n^2(x^\perp, t)dt^2 \\ &+ a^2(x^\perp, t)\left(\delta_{ij} + k\frac{x_i x_j}{1 - kr^2}\right)dx^i dx^j + b^2(x^\perp, t)dx^{\perp 2} \end{aligned} \tag{59}$$

can be found in [14]. Brane cosmology in different backgrounds or with different *ansätzen* for the metric has been a subject of numerous studies. Investigations of cosmology in backgrounds motivated by M-theory or generalizations of the Randall–Sundrum model can be found in [81, 15, 73, 94, 32, 26, 24, 51, 76, 80, 110, 33, 90, 69, 52, 88, 84, 27, 72, 95, 16, 25, 29]. Eq. (59) implies a brane cosmological principle in that it presupposes that every hypersurface $x^\perp = \text{const.}$ is a Robertson–Walker spacetime with cosmological time $T|_{x^\perp} = \int n(x^\perp, t)dt$.

I will focus on the cosmological aspects of the model of Dvali, Gabadadze and Porrati. Building on the results of [15, 14], the evolution equations of a 3-brane in a five-dimensional bulk following from (36) and (37) were so neatly presented in recent papers by Deffayet [36] and by Deffayet, Dvali and Gabadadze [37] that I decided to give the corresponding results for a ν-brane¶.

The Einstein tensors for the metric (59) in Gaussian normal coordinates ($b^2 = 1$) and in $d = \nu + 1$ spatial dimensions are
on the hypersurfaces $x^\perp = \text{const.}$:

$$G_{00}^{(\nu)} = \frac{1}{2}\nu(\nu-1)n^2\left(\frac{\dot{a}^2}{n^2a^2} + \frac{k}{a^2}\right) \tag{60}$$

$$G_{ij}^{(\nu)} = (\nu-1)\left(\frac{\dot{n}\dot{a}}{n^3a} - \frac{\ddot{a}}{n^2a}\right)g_{ij} - \frac{1}{2}(\nu-1)(\nu-2)\left(\frac{\dot{a}^2}{n^2a^2} + \frac{k}{a^2}\right)g_{ij}, \tag{61}$$

and in the bulk:

$$G_{00} = \frac{1}{2}\nu(\nu-1)n^2\left(\frac{\dot{a}^2}{n^2a^2} - \frac{a'^2}{a^2} + \frac{k}{a^2}\right) - \nu n^2\frac{a''}{a}, \tag{62}$$

$$\begin{aligned} G_{ij} = {} & \frac{1}{2}(\nu-1)(\nu-2)\left(\frac{a'^2}{a^2} - \frac{\dot{a}^2}{n^2a^2} - \frac{k}{a^2}\right)g_{ij} \\ & + (\nu-1)\left(\frac{a''}{a} + \frac{n'a'}{na} - \frac{\ddot{a}}{n^2a} + \frac{\dot{n}\dot{a}}{n^3a}\right)g_{ij} + \frac{n''}{n}g_{ij}, \end{aligned} \tag{63}$$

$$G_{0\perp} = \nu\left(\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a}\right), \tag{64}$$

$$G_{\perp\perp} = \frac{1}{2}\nu(\nu-1)\left(\frac{a'^2}{a^2} - \frac{\dot{a}^2}{n^2a^2} - \frac{k}{a^2}\right) + \nu\left(\frac{n'a'}{na} + \frac{\dot{n}\dot{a}}{n^3a} - \frac{\ddot{a}}{n^2a}\right). \tag{65}$$

The matching conditions (37) for an ideal fluid on the brane

$$T_{00} = \varrho n^2, \; T_{ij} = pg_{ij}$$

read

$$\begin{aligned} \lim_{\epsilon\to+0}[\partial_\perp n]_{x^\perp=-\epsilon}^{x^\perp=\epsilon} = {} & \frac{n}{\nu m_{\nu+1}^\nu}\Big((\nu-1)\varrho + \nu p\Big)\Big|_{x^\perp=0} \\ & + \frac{m_\nu^{\nu-1}}{m_{\nu+1}^\nu}(\nu-1)n\left(\frac{\ddot{a}}{n^2a} - \frac{\dot{a}^2}{2n^2a^2} - \frac{\dot{n}\dot{a}}{n^3a} - \frac{k}{2a^2}\right)\Big|_{x^\perp=0}, \end{aligned} \tag{66}$$

¶ This moderate generalization spares me from the frustrating experience of plainly repeating the equations of Deffayet *et al.* It also may be of some interest in its own to have the corresponding equations for a $(\nu+1)$-dimensional timelike hypersurface at hand.

$$\lim_{\epsilon\to+0}\left[\partial_\perp a\right]_{x^\perp=-\epsilon}^{x^\perp=\epsilon}=\frac{m_\nu^{\nu-1}}{2m_{\nu+1}^\nu}(\nu-1)\left(\frac{\dot{a}^2}{n^2a}+\frac{k}{a}\right)\bigg|_{x^\perp=0}-\frac{\varrho a}{\nu m_{\nu+1}^\nu}\bigg|_{x^\perp=0}. \quad (67)$$

In the spirit of the remark following eq. (37) this corresponds to effective gravitational contributions to the pressure and energy density on the brane:

$$\varrho_G=-\frac{1}{2}\nu(\nu-1)m_\nu^{\nu-1}\left(\frac{\dot{a}^2}{n^2a^2}+\frac{k}{a^2}\right),$$

$$p_G=(\nu-1)m_\nu^{\nu-1}\left(\frac{\ddot{a}}{n^2a}-\frac{\dot{n}\dot{a}}{n^3a}\right)+\frac{1}{2}(\nu-1)(\nu-2)m_\nu^{\nu-1}\left(\frac{\dot{a}^2}{n^2a^2}+\frac{k}{a^2}\right).$$

Not surprisingly, energy conservation on the brane follows from the absence of transverse momentum, $T_{0\perp}=0$. With (64) this implies

$$\frac{n'}{n}=\frac{\dot{a}'}{\dot{a}} \quad (68)$$

and in particular

$$\lim_{\epsilon\to+0}\left[\frac{n'}{n}\right]_{x^\perp=-\epsilon}^{x^\perp=\epsilon}=\lim_{\epsilon\to+0}\left[\frac{\dot{a}'}{\dot{a}}\right]_{x^\perp=-\epsilon}^{x^\perp=\epsilon}.$$

Insertion of (66,67) into this equation yields the sought for conservation equation

$$\dot{\varrho}a\Big|_{x^\perp=0}=-\nu(\varrho+p)\dot{a}\Big|_{x^\perp=0}. \quad (69)$$

Insertion of (68) into (62) and (65) for $x^\perp\neq 0$ yields a ν-dimensional version of the integral of Binétruy *et al.* [14]:

$$\frac{2}{\nu n^2}a'a^\nu G_{00}=\frac{\partial}{\partial x^\perp}\left(\frac{\dot{a}^2}{n^2}a^{\nu-1}-a'^2a^{\nu-1}+ka^{\nu-1}\right)=0,$$

$$\frac{2}{\nu}\dot{a}a^\nu G_{\perp\perp}=-\frac{\partial}{\partial t}\left(\frac{\dot{a}^2}{n^2}a^{\nu-1}-a'^2a^{\nu-1}+ka^{\nu-1}\right)=0,$$

i.e.

$$I_{BDEL}^+=\left(\frac{\dot{a}^2}{n^2}-a'^2+k\right)a^{\nu-1}\bigg|_{x^\perp>0} \quad (70)$$

and

$$I_{BDEL}^-=\left(\frac{\dot{a}^2}{n^2}-a'^2+k\right)a^{\nu-1}\bigg|_{x^\perp<0} \quad (71)$$

are two constants, with $I_{BDEL}^+=I_{BDEL}^-$ if

$$\lim_{\epsilon\to+0}a'\Big|_{x^\perp=\epsilon}=\pm\lim_{\epsilon\to+0}a'\Big|_{x^\perp=-\epsilon}.$$

We have not yet taken into account $G_{ij}=0$ in the bulk. However, eq. (68) implies $\partial_\perp(n/\dot{a})=0$, and therefore

$$\frac{n''}{n}=\frac{\dot{a}''}{\dot{a}}.$$

This, the bulk equations $G_{00}=G_{\perp\perp}=0$, and the constancy of $I^\pm$ imply that the bulk equation $G_{ij}=0$ is already satisfied and does not provide any new information.

We can now simplify the previous equations by further restricting our Gaussian normal coordinates through the gauge

$$n(0,t)=1 \tag{72}$$

by simply performing the transformation

$$t\ \Rightarrow\ t_{FRW}=\int^{t}dt'\,n(0,t')$$

of the time coordinate. This gauge is convenient because it gives the usual cosmological time on the brane. Henceforth this gauge will be adopted, but the index FRW will be omitted.

Eqs. (68,72) imply that our set of dynamical variables is not $\{n(x^{\perp},t),a(x^{\perp},t)\}$ but only $a(x^{\perp},t)$, with $n(x^{\perp},t)$ given by

$$n(x^{\perp},t)=\frac{\dot{a}(x^{\perp},t)}{\dot{a}(0,t)}.$$

The basic set of cosmological equations in the present setting (without a cosmological constant in the bulk) are thus eqs. (67,69,70,71), which have to be amended with dispersion relations (or corresponding evolution equations) for the ideal fluid components on the brane:

$$\boxed{\begin{aligned}
&\lim_{\epsilon\to+0}\left[\partial_{\perp}a\right]_{x^{\perp}=-\epsilon}^{x^{\perp}=\epsilon}(t)=\frac{m_{\nu}^{\nu-1}}{2m_{\nu+1}^{\nu}}(\nu-1)\frac{\dot{a}^{2}(0,t)+k}{a(0,t)}-\frac{\varrho(t)a(0,t)}{\nu m_{\nu+1}^{\nu}},\\
&I_{BDEL}^{\pm}=\left(\dot{a}^{2}(0,t)-a'^{2}(x^{\perp},t)+k\right)a^{\nu-1}(x^{\perp},t)\Big|_{x^{\perp}\gtrless 0},\\
&\dot{\varrho}(t)a(0,t)=-\nu(\varrho(t)+p(t))\dot{a}(0,t),\\
&p(t)=p(\varrho(t)).
\end{aligned}}$$

Our primary concern with regard to observational consequences is the evolution of the scale factor $a(0,t)$ on the brane, and we can use *les intégrales françaises* $I^{\pm}$ to eliminate the normal derivatives $a'(x^{\perp}\to\pm0,t)$ from the brane analogue of the Friedmann equation:

$$\begin{aligned}
&\pm\sqrt{\dot{a}^{2}(0,t)+k-I_{BDEL}^{+}a^{1-\nu}(0,t)}\mp\sqrt{\dot{a}^{2}(0,t)+k-I_{BDEL}^{-}a^{1-\nu}(0,t)}\\
&=\frac{m_{\nu}^{\nu-1}}{2m_{\nu+1}^{\nu}}(\nu-1)\frac{\dot{a}^{2}(0,t)+k}{a(0,t)}-\frac{\varrho(t)a(0,t)}{\nu m_{\nu+1}^{\nu}}.
\end{aligned} \tag{73}$$

If this equation is solved for $a(0,t)$ by using the dispersion relation and energy conservation on the brane, then $a(x^{\perp},t)$ can be determined in the bulk from the constancy of $I^{\pm}$.

There must be at least one minus sign on the left hand side of (73) if the right hand side is negative, but the dynamics of the problem does not require symmetry across the brane. The constants $I^{\pm}$ must be considered as initial conditions, and if e.g. $I^+ \neq I^-$, then there cannot be any symmetry across the brane.

If $m_\nu \neq 0$ and the normal derivatives on the brane have the same sign:

$$m_\nu a'(x^\perp \to +0, t) a'(x^\perp \to -0, t) > 0, \tag{74}$$

then the cosmology of our brane approximates ordinary Friedmann–Robertson–Walker cosmology during those epochs when

$$I^{\pm}_{BDEL} \ll \left(\dot{a}^2(0,t) + k\right) a^{\nu-1}(0,t).$$

In particular, this applies to late epochs in expanding open or flat branes ($k \neq 1$).

However, standard cosmology may be realized in this model in an even more direct way: If (74) holds and $I^+ = I^-$, then (73) reduces *entirely* to the ordinary Friedmann equation for a $(\nu+1)$-dimensional spacetime. The evolution of the background geometry of the observable universe according to the Friedmann equation can thus be embedded in the model of Dvali, Gabadadze and Porrati, with the bevavior of $a(x^\perp, t)$ off the brane determined solely by the integral $I^+ = I^-$ and the boundary condition $a(0,t)$ from the Friedmann equation.

This possibility of a direct embedding of Friedmann cosmology is a consequence of the fact that the evolution of the background geometry (59) and the source terms ϱ, p are supposed to depend only on t and $x^\perp$. This implies the possibility to decouple the brane and the bulk contributions in the Einstein equation for the background metric, and in this case deviations from Friedmann–Robertson–Walker cosmology would only show up in specific $\mathbf{x}$-dependent effects like the evolution of cosmological perturbations and structure formation.

As a simple example of the realization of this direct embedding of a Friedmann–Robertson–Walker 3-brane in the model of Dvali *et al.* we consider a spatially flat ($k = 0$) radiation dominated ($p = \varrho/3$) 3-brane with continuous normal derivative a' across the brane:

Since a is smooth across the brane we have $I^+ = I^-$ and the signs in (73) conspire in such a way that the left hand side vanishes. With $\nu = 3$ the right hand side boils down to the ordinary Friedmann equation in a spatially flat radiation dominated background, with solution

$$\begin{aligned} \varrho(t) &= \frac{3m_3^2}{4t^2}, \\ a(0,t) &= C\sqrt{t}. \end{aligned} \tag{75}$$

The integral of Binétruy *et al.* then yields the differential equation

$$a^2(x^\perp, t) a'^2(x^\perp, t) + I_{BDEL} = \frac{C^2}{4t} a^2(x^\perp, t),$$

which has to be solved under the boundary condition (75). This yields

$$a^2(x^\perp, t) = \frac{C^2}{4t} x^{\perp 2} + \sqrt{C^4 - 4I_{BDEL}}\, x^\perp + C^2 t$$

and

$$n^2(x^\perp, t) = \frac{C^2}{4t^2} \frac{\left(4t^2 - x^{\perp 2}\right)^2}{C^2 x^{\perp 2} + 4\sqrt{C^4 - 4I_{BDEL}}\, x^\perp t + 4C^2 t^2}.$$

There is a coordinate singularity on the spacelike hypersurfaces $x^\perp = \pm 2t$, which indicates that the orthogonal geodesics emerging from the 3-brane do not cover the full five-dimensional manifold, but our Gaussian normal coordinates were anyhow expected to cover only a neighborhood of the brane.

The possibility to describe the cosmological evolution of the 3-brane background geometry by an ordinary Friedmann equation implies that we will have to rely on specifically $\mathbf{x}$-dependent effects to observationally distinguish Friedmann cosmology from brane cosmology.

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