How much canonical are Ashtekar's variables ?

Vladimir O. Soloviev*

Institute for High Energy Physics

Protvino, Moscow Region, 142 284, Russia

Abstract

Attention is paid to the fact that in field theory a commutator of functional derivatives may differ from zero by surface integrals. Ashtekar's formalism is a primer which demonstrates that transformations looking locally as canonical, can lead to the appearance of surface terms in symplectic form of a field theory. The prescription for δ-function is given that allows to preserve surface terms in local calculations of Poisson brackets.

*E-Mail: vosoloviev@M10.IHEP.SU

Recently proposed new Hamiltonian variables[1] are of great interest in connection with the possibilities that they open in quantum gravity.[2,3] In a set of papers[4−7] it is shown that Ashtekar's variables arise after a sequence of canonical transformations from the standard tetrad Hamiltonian formalism. The keypoint of this sequence is the transformation to the complex (anti)self-dual connection. We will show that just in this transformation some surface terms were ignored, though these terms are not zero, for example, in asymptotically flat space under the boundary conditions accepted in publications.[8,9] The reason was that some statements proved only in mechanics were extrapolated to field theory. At the same time, the Ashtekar formalism will be considered as a primer in which we see entering of surface integrals into symplectic form of field theory.

If in mechanics the relation

$$\frac{\partial^2 F(q)}{\partial q^A \partial q^B} = \frac{\partial^2 F(q)}{\partial q^B \partial q^A}, \tag{1}$$

is an identity, then in field theory for the analogous equality

$$\frac{\delta^2 F[\phi]}{\delta\phi^A(x)\delta\phi^B(y)} = \frac{\delta^2 F[\phi]}{\delta\phi^B(y)\delta\phi^A(x)} \tag{2}$$

it is not, in general, true because of presence of partial derivatives in the integrand of $F[\phi]$, where for simplicity only first derivatives are considered

$$F[\phi] = \int\limits_V f\left(\phi^C(z^i), \frac{\partial\phi^C(z^i)}{\partial z^j}\right) d^n z, \tag{3}$$

and the coordinate invariant integral is taken over a connected finite or infinite region V with a smooth boundary ∂V (A, B, C are some abstract indexes). This may look nonevident if operate with δ–functions and the reason lies in the inadequacy of this formalism. We shall comment upon this below. But it is easy to do the job by taking smeared functional derivatives defined as

$$\int\limits_V N^A(x)\frac{\delta F(\phi)}{\delta\phi^A(x)} d^n x = \int\limits_V N^A(x)\left(\frac{\partial f}{\partial\phi^A} - \frac{\partial}{\partial x^i}\frac{\partial f}{\partial\phi^A_{,i}}\right) d^n x, \tag{4}$$

where $N^A(x)$ is an arbitrary infinitely differentiable function with transformational properties preserving the coordinate invariance of the integral. Then we can see

$$\int\limits_V d^n x \int\limits_V d^n y N^A(x) M^B(y) \left[\frac{\delta^2}{\delta\phi^B(y)\delta\phi^A(x)} - \frac{\delta^2}{\delta\phi^A(x)\delta\phi^B(y)}\right] F[\phi] =$$

$$= \oint\limits_{\partial V} (N^A M^B - N^B M^A) \frac{\partial^2 f}{\partial\phi^{[B}\partial\phi^{A]}_{,j}} dS_j + \oint\limits_{\partial V} (N^B M^A_{,i} - M^B N^A_{,i}) \frac{\partial^2 f}{\partial\phi^B_{,j}\partial\phi^A_{,i}} dS_j, \quad (5)$$

where surface integrals are taken over the boundary ∂V, the square brackets denote antisymmetrization and comma– a coordinate partial derivative.

Now let us trace the sequence of transformations which lead to Ashtekar variables. The symplectic form of the tetrad Hamiltonian formalism of gravitation in the time gauge is

$$\Omega = \int \delta E^a_i(x) \wedge \delta\pi^{ia}(x) d^3x, \quad (6)$$

where in our notations, the triad indices are: $a, b, c, ...$; and the spatial ones - $i, j, k, ...$. As was shown[4] earlier these variables first should be changed for

$$\tilde{E}^{ia} = E E^{ia}, \qquad K^a_i(x) = K_{ij} E^{ja} + E^{-1} J^{ab} E_{ib}, \quad (7)$$

where E^{ia} is the inverse matrix of E^a_i, $E = \det(E^a_i)$, K_{ij} is the second fundamental form of the time hypersurface and J^{ab} are the six constraints generating triad rotations. Then we have

$$\{\tilde{E}^{ia}(x), K^b_j(y)\} = \frac{1}{2}\delta^i_j \delta^{ab} \delta(x,y), \qquad \{K^a_i(x), K^b_j(y)\} = 0. \quad (8)$$

The final step to Ashtekar variables is

$$\begin{aligned} \tilde{E}^{ia}(x) &\rightarrow \tilde{E}^{ia}(x), \\ K^a_i(x) &\rightarrow {}^{\pm}A^a_i(x) = \pm i K^a_i(x) + \frac{\delta}{\delta\tilde{E}^{ia}(x)} F[\tilde{E}], \end{aligned} \quad (9)$$

where we are free to use ${}^{+}A^a_i(x)$ or ${}^{-}A^a_i(x)$ as a conjugate to $\tilde{E}^{ia}$,

$$F[\tilde{E}] = \int \tilde{E}^{jb}(y) \Gamma^b_j(y) d^3y, \qquad \Gamma^a_i(x) = \frac{1}{2}\epsilon^{abc}(E_{jc}\partial_i E^{jb} + \Gamma^k_{ij} E^{jb} E_{kc}), \quad (10)$$

Γ^k_{ij} are Christoffel symbols and

$$\frac{\delta F[\tilde{E}]}{\delta \tilde{E}^{ia}(x)} = \Gamma^a_i(x), \tag{11}$$

because terms with $\delta\Gamma^a_i$ are divergences and give no contribution to the functional derivative. The Poisson bracket

$$\{ \int_V N^{ia}(x)^{\pm} A^a_i(x) d^3x, \int_V M^{jb}(y)^{\pm} A^b_j(y) d^3y \} =$$

$$= \pm\frac{i}{2} \int_V d^3x \int_V d^3y N^{ia}(x) M^{jb}(y) \left[\frac{\delta}{\delta \tilde{E}^{jb}(y)} \frac{\delta}{\delta \tilde{E}^{ia}(x)} - \frac{\delta}{\delta \tilde{E}^{ia}(x)} \frac{\delta}{\delta \tilde{E}^{jb}(y)} \right] F[\tilde{E}], \tag{12}$$

according to (5) is different from zero. Therefore the Poisson brackets in Ashtekar formalism should be defined by formulae:

$$\{H_1, H_2\} = \int_V d^3x \int_V d^3y [\left(\frac{\delta H_1}{\delta \tilde{E}^{ia}(x)} \frac{\delta H_2}{\delta^{\pm} A^b_j(y)} - \frac{\delta H_2}{\delta \tilde{E}^{ia}(x)} \frac{\delta H_1}{\delta^{\pm} A^b_j(y)} \right) \times$$
$$\times \{ \tilde{E}^{ia}(x), {}^{\pm}A^b_j(y) \} + \frac{\delta H_1}{\delta^{\pm} A^a_i(x)} \frac{\delta H_2}{\delta^{\pm} A^b_j(y)} \{ {}^{\pm}A^a_i(x), {}^{\pm}A^b_j(y) \}], \tag{13}$$

or

$$\{H_1, H_2\} = \pm\frac{i}{2} \int_V \left[\frac{\delta H_1}{\delta \tilde{E}^{ia}} \frac{\delta H_2}{\delta^{\pm} A^a_i} - \frac{\delta H_2}{\delta \tilde{E}^{ia}} \frac{\delta H_1}{\delta^{\pm} A^a_i} \right] d^3x \pm$$

$$\pm\frac{i}{4} \oint_{\partial V} \epsilon^{acd} E^{-1} (\delta^k_j \delta^{bd} E_{ic} - E_{ib} E_{jc} E^{kd}) \left[\frac{\delta H_1}{\delta^{\pm} A^a_i} \frac{\delta H_2}{\delta^{\pm} A^b_j} - \frac{\delta H_2}{\delta^{\pm} A^a_i} \frac{\delta H_1}{\delta^{\pm} A^b_j} \right] dS_k, \tag{14}$$

and the corresponding symplectic form is

$$\Omega = \pm 2i \int_V \delta^{\pm} A^a_i \wedge \delta \tilde{E}^{ia} d^3x \pm 4i \oint_{\partial V} \epsilon^{acd} E^{-1} (\delta^k_j \delta^{bd} E_{ic} - E_{ib} E_{jc} E^{kd}) \delta \tilde{E}^{ia} \wedge \delta \tilde{E}^{jb} dS_k. \tag{15}$$

Of course, calculations made for compact spacetime sections are not affected by these redefinitions. An example of crucial importance of these corrections is the canonical realization of Poincaré algebra arising at spatial infinity in asymptotically flat spacetime. In the standard metric canonical formalism of gravity the Poincaré group generators are expressed in terms of surface integrals taken over a two-dimensional sphere at spatial infinity, as was proved for the first time by Regge and Teitelboim[1]0. These surface integrals play two roles : they guarantee differentiability of the Hamiltonian at the prescribed boundary conditions and they also enter the Poisson brackets algebra

$$\{\, H(N^{\alpha}), H(M^{\beta})\} \approx H([N,M]^{\gamma}), \tag{16}$$

where the square brackets denote the commutator of vector fields (the equality is maintained in the "strong sense" only for surface integrals when each vector field represents coordinate transformation from the asymptotic Poincaré group, because different expression than $[N,M]^{\gamma}$ is present in the "algebra" of constraints[11,12]). The second role was discussed much less than the first one. The criterion of differentiability that is usually exploited can give surface terms only up to the phase space constants. But the straightforward calculation of all the surface terms in the Poisson brackets[13] has not such shortcoming(see also a footnote in Arnold's book[14] though not touching surface integrals) and gives us many advantages, for example, a possibility to search for new boundary conditions or to evaluate central charges. Later this method was mentioned[15] with a remark that "such a calculation is typically very cumbersome". Nevertheless the same authors have proved in other publication[16] that a Poisson bracket of two differentiable in Regge-Teitelboim's sense generators is also a differentiable generator.

We have shown recently[17] that without redefinition of the Poisson brackets the realization of the algebra of Poincaré group generators in the Ashtekar's formalism for the Schwarzschild solution meets serious difficulties. But with the new Poisson brackets (14) the algebra of Poincaré group generators can be definitely realized[18] for the Schwarzschild solution and the generators have values that coincide with Ashtekar's expressions.[8,9]

Now let us give the promised comment on the inadequacy of usual handling of δ-functions for the problem of calculating surface terms in canonical Poisson brackets. If first the brackets of local expressions for integrands are calculated then to obtain the resulting integral form it is necessary to evaluate terms like this,

$$\int\limits_V d^n x \int\limits_V d^n y \phi(x)\psi(y) \frac{\partial^{(k)}}{\partial x^{i_1}\ldots\partial x^{i_k}} \frac{\partial^{(l)}}{\partial y^{j_1}\ldots\partial y^{j_l}} \delta(x-y). \tag{17}$$

But such integrals are defined in the standard theory of distributions[19] only when the smooth (C^∞) test functions ϕ, ψ have their support strictly inside the region V, that is, when no surface integrals can appear at all. In other case surface terms that arise after usual formal manipulations with the δ-functions are ambiguous and depend upon order of integration. It is not difficult to see that the result will be unique and just the same as that obtained through straightforward integration of Eulerian derivatives by the standard formula:

$$F, G = \int_V \left(\frac{\delta F}{\delta q^A(x)} \frac{\delta G}{\delta p_A(x)} - \frac{\delta G}{\delta q^A(x)} \frac{\delta F}{\delta p_A(x)} \right) d^n x, \tag{18}$$

if we define (going slightly outside the standard theory of distributions, but not pretending to propose a new one) the integral (17) as

$$(-1)^{k+l} \int_V \frac{\partial^{(k)} \phi}{\partial x^{i_1} \ldots \partial x^{i_k}} \frac{\partial^{(l)} \psi}{\partial x^{j_1} \ldots \partial x^{j_l}} d^n x. \tag{19}$$

In other words, a derivative of the δ-function, taken over one argument, should be transported to the test function of just the same argument. Therefore, the ambiguity in local calculations arises because of unjustified using the relation

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) \delta(x, y) = 0. \tag{20}$$

It is not difficult to check that the surface integrals in Poisson brackets will coincide with those obtained on the base of Eulerian derivatives from (18) if instead of (20) the following formula is applied:

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) \delta(x, y) = -\delta_i(S_x) \delta(x, y), \tag{21}$$

where the surface δ-function is defined very similarly to[19] as

$$\int_V f^i(x) \delta_i(S_x) d^n x = \oint_{\partial V} f^i(x) dS_i \equiv \int_V f^i_{,i}(x) d^n x, \tag{22}$$

and $f^i(x)$ should be a vector density, infinitely differentiable (C^∞) in an n-dimensional region V together with its boundary ∂V. All usual formulae for the δ -function and its derivatives taken over only one of its arguments are applicable. By differentiating (21) useful formulae for mixed second and third derivatives can be deduced

$$\begin{aligned}
\frac{\partial^2}{\partial x^i \partial y^j}\delta(x,y) &= -\frac{\partial^2}{\partial x^i \partial x^j}\delta(x,y) - \delta_j(S_y)\frac{\partial}{\partial x^i}\delta(x,y) = \\
&= -\frac{\partial^2}{\partial y^i \partial y^j}\delta(x,y) - \delta_i(S_x)\frac{\partial}{\partial y^j}\delta(x,y), \qquad (23)\\
\frac{\partial^3}{\partial x^i \partial x^j \partial y^k}\delta(x,y) &= -\frac{\partial^3}{\partial x^i \partial x^j \partial x^k}\delta(x,y) - \delta_k(S_y)\frac{\partial^2}{\partial x^i \partial x^j}\delta(x,y). \qquad (24)
\end{aligned}$$

Really, these rules are enough for evaluating surface integrals in Poisson brackets in usual situations, for example, in the metric Hamiltonian formalism of general relativity where the answer is known.[13]

The main conclusion the author would like to make, refers to generality of the obtained result. The noncommutativity of the variational derivatives considered here should have a wide range of application and through a new light on canonical transformations in field theory. Such transformations are usually inherent in the procedure of reduction in gauge theories. We plan to discuss this point in a separate paper. It seems to us extremely interesting and important that in Ashtekar's formalism we see the appearance of surface integrals in the symplectic form of field theory. Regge and Teitelboim taught us how to deal with surface integrals arising in the Hamiltonian. They suggested that the freedom to add surface terms must be limited by the requirement of absence of any surface integrals in δH. The presence of surface terms in a symplectic form , probably, forces us to extend the phase space and to include boundary values into it as new canonical variables. These surface variables should have their own Hamiltonian and how to find it we should be taught by a new approach.

References

[1] A.Ashtekar, *Phys. Rev. Lett.* **57** (1986) 2244.

[2] T.Jacobson, L.Smolin, *Nucl. Phys.* **B299** (1988) 295.

[3] C.Rovelli, L.Smolin, *Nucl. Phys.* **B331** (1990) 80.

[4] M.Henneaux, J.E.Nelson, C.Schomblond, *Phys. Rev.* **D39** (1989) 434.

[5] J.N.Goldberg, *Phys. Rev.* **D37** (1988) 2116.

[6] J.L.Friedman, I.Jack, *Phys. Rev.* **D37** (1988) 3495.

[7] B.P.Dolan, *Phys. Lett.* **B233** (1989) 89.

[8] A.Ashtekar, *Phys. Rev.* **D36** (1987) 1587.

[9] A.Ashtekar, P.Mazur, C.G.Torre, *Phys. Rev.* **D36** (1987) 2955.

[10] T.Regge, C.Teitelboim, *Ann. of Phys.* **88** (1974) 286.

[11] P.G.Bergmann, A.Komar, *Intern. J. Theor. Phys.* **5** (1972) 15.

[12] C.Rovelli, *Nuovo Cim.* **B92** (1986) 49.

[13] V.O.Soloviev, *Teor. Mat. Fiz.* **65** (1985) 400.

[14] V.I.Arnold, “Mathematical methods of classical mechanics”, Moscow, “Nauka”, 1974 (in Russian), page 186.

[15] J.D.Brown, M.Henneaux, *Commun. Math. Phys.* **104** (1986) 207.

[16] J.D.Brown, M.Henneaux, *J. Math. Phys.* **27** (1986) 489.

[17] V.O.Soloviev (to appear in the MG6 Proceedings, Kyoto, 1991).

[18] V.O.Soloviev, “Poincaré algebra in Ashtekar’s formalism”, Preprint IHEP, Protvino, 1992 (in preparation).

[19] V.S.Vladimirov, “Equations of mathematical physics”, Moscow, “Nauka”, 1967 (in Russian).