POISSON BRACKETS FOR TOTAL DIVERGENCES

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ABSTRACT

The standard Poisson brackets in field theory do not fulfil the Jacobi identity if boundary values are not reasonably fixed by boundary conditions. We show that these brackets can be modified by adding some surface terms to avoid this restriction. Our definition of Poisson brackets is applicable to any local form of them, permits to treat boundary values of a field on equal footing with its internal values and directly estimate the brackets between both surface and volume integrals.

The field theory Hamiltonian formalism have been paid a lot of attention in the two previous decades due to studying of integrable models. Now the concepts of Poisson bracket and Hamiltonian operator are deeply elaborated, see for example Refs. 1,2.

But from physicist's viewpoint the treatment based on free integration by parts sometimes seems too restrictive. There are problems where surface integrals are nonzero and have physical meaning. We can mention characteristics of the gravitational field in asymptotically flat space-times,[3] surface energy of a fluid,[4] etc. As a result, in the aforementioned works the concept of Poisson bracket is exploited outside its definition given in Refs. 1,2 and in many other recent mathematical textbooks.

In this report it is proposed to generalize the definition of Poisson brackets in order to admit treatment of divergences as normal Hamiltonian densities that generate a kind of Hamiltonian equations on the boundary.

We use the local coordinate language and consider a domain Ω in R^n having a smooth boundary $\partial\Omega$. The characteristic function of this domain is $\theta_\Omega = \theta(P_\Omega)$, where equation $P_\Omega(x^1, ..., x^n) = 0$ defines the boundary.

Definition 1

An integral over a finite domain Ω of a function of field variables $\phi^A(x), A = 1, ..., p$ and their partial derivatives $D_J\phi^A$ up to some finite order

$$F = \int_\Omega f(\phi_A(x), D_J\phi_A(x))d^n x \tag{1}$$

is called a local functional.

All the functions f and ϕ_A as well as their variations are supposed to be infinitely smooth, i.e. $C^\infty(R^n)$. We use the multi-index notations $J = (j_1, ..., j_n)$

$$D_J = \frac{\partial^{|J|}}{\partial^{j_1} x^1 ... \partial^{j_n} x^n}, \qquad |J| = j_1 + ... + j_n. \tag{2}$$

Binomial coefficients for multi-indices are

$$\binom{J}{K} = \binom{j_1}{k_1} \cdots \binom{j_n}{k_n}, \tag{3}$$

We denote as $\mathcal{A}$ the space of local functionals . It is very important that this space includes functionals with integrands depending on derivatives of arbitrary order. Otherwise the Poisson brackets could go out of $\mathcal{A}$.

Definition 2

A bilinear operation $\{\cdot,\cdot\}$ such that for any $F, G, H \in \mathcal{A}$

1) $\{F,G\} \in \mathcal{A}$;

2) $\{F,G\} = -\{G,F\}$;

3) $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$;

is called the new field theory Poisson bracket.

The standard[1,2] definitions of local functional and Poisson bracket differs only by adding words "modulo divergences".

It occurs possible to construct new expressions for field theory Poisson brackets fulfilling the new definition, by supposing locality and antisymmetry, i.e.

$$\{\phi_A(x), \phi_B(y)\} = \frac{1}{2} \sum_L \left(I^L_{AB}(x) D^{(x)}_L - I^L_{BA}(y) D^{(y)}_L \right) \delta(x,y), \tag{4}$$

and these expressions are the same for all kinds of boundary conditions.

$$\frac{1}{2} \sum_{A,B,P,Q} \int_\Omega D_{P+Q} \left(E^P_A(f) \hat{I}_{AB} E^Q_B(g) - E^P_A(g) \hat{I}_{AB} E^Q_B(f) \right) d^n x, \tag{5}$$

where

$$\hat{I}_{AB} = \sum_N I^N_{AB} D_N, \tag{6}$$

and coefficients I^N_{AB} should satisfy some standard conditions[5]. Higher Eulerian operators E^J_A are defined in Ref. 1 through the formula of the full variation of a local functional

$$\delta F = \sum_{A,J} \int_\Omega D_J \left(E^J_A(f) \delta\phi_A \right) d^n x. \tag{7}$$

They can be explicitly expressed as

$$E^J_A(f) = \sum_K (-1)^{|K|+|J|} \binom{K}{J} D_{K-J} \frac{\partial f}{\partial \phi^{(K)}_A}. \tag{8}$$

Usual variational derivative (or Euler-Lagrange derivative) is the Eulerian operator of zeroth order. Let us mention that if J is not included into K, then all quantities having multi-index $(K-J)$ are zero. The sums over J in and K are really finite because a local functional can depend only on a finite number of derivatives.

The change of the old brackets for the new ones can be understood as a transition from functional variations with fixed values of the field on the boundary to free variations. Then a natural construction that we call a full variational derivative appears in the place of the standard Euler-Lagrange variational derivative. This new object cares information not only on the integrand (up to constants), but also on the domain of integration Ω

$$\frac{\delta F}{\delta \phi_A} = E^0_A(\theta_\Omega f) = \sum_J (-1)^{|J|} E^J_A(f) D_J \theta_\Omega, \tag{9}$$

where θ_Ω is a characteristic function of the domain.

Inserting this variational derivative into the standard formula

$$\{F, G\} = \sum \int\int \frac{\delta F}{\delta \phi_A(x)} \{\phi_A(x), \phi_B(y)\} \frac{\delta G}{\delta \phi_B(y)} d^n x d^n y, \tag{10}$$

we should formally integrate by parts separately in x and y variables to remove derivatives from the δ-function. Afterwards we take off one of the integrations naively with the help of δ-function and obtain an integral containing products of distributions. By mystical way it occurs that using a formal rule

$$D_J \theta(P_\Omega) \times D_K \theta(P_\Omega) = D_{J+K} \theta(P_\Omega) \tag{11}$$

to transform integrand into the legal form, and then integrating by parts over infinite region we get a formula (5) that satisfies the new definition of Poisson brackets given above.

Evidently to prove this statement we should prove Jacobi identity and for wide classes of local brackets this is done in Ref. 5.

Let us now consider a simple example of the Hamiltonian system and boundary equations obtained through these new brackets. The nonlinear Schrödinger equation can be treated as arising from the Hamiltonian

$$H = 1/2 \int (\mathcal{H} + \bar{\mathcal{H}}) dx, \tag{12}$$

where

$$\mathcal{H} = r' q' + k r^2 q^2, \qquad \bar{\mathcal{H}} = \bar{r}' \bar{q}' + k \bar{r}^2 \bar{q}^2, \tag{13}$$

and Poisson brackets

$$\{q(x), r(y)\} = -2i\delta(x, y), \qquad \{\bar{q}(x), \bar{r}(y)\} = 2i\delta(x, y). \tag{14}$$

To return to the standard notations we should put reality conditions

$$\psi = q = \bar{r}, \qquad \bar{\psi} = r = \bar{q}. \tag{15}$$

If we now consider Poisson brackets with the Hamiltonian for integrals of the full space derivatives of canonical variables $\phi_A = (q, r)$, we will obtain dynamical equations on the boundary

$$\{\int {\phi'}_A dx, H\} = \dot{\phi}_A(2) - \dot{\phi}_A(1) = I_{AB} \frac{\partial \mathcal{H}}{\partial \phi_B}\Big|_1^2, \tag{16}$$

which are different from standard (internal) equations

$$\dot{\phi_A} = I_{AB} E^0_B(\mathcal{H}) \tag{17}$$

For the given Hamiltonian we have

$$\dot{q} = -4ikrq^2, \qquad \dot{r} = 4ikr^2q. \tag{18}$$

These equations can be easily integrated and give elementary oscillations at the ends:

$$\psi_{boundary}(t) = \psi_{boundary}(0)exp(-4ik|\psi_{boundary}(0)|^2t). \tag{19}$$

We can see that in this case the dynamics of boundary values is separated from the internal one. Of course, this situation is not general.

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