SURFACE INTEGRALS OF POINCARÉ ALGEBRA IN ASHTEKAR'S FORMALISM

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ABSTRACT

When an asymptotically flat spacetime is considered the new canonical formulation of General Relativity invented by Ashtekar requires a due account of surface integrals which are necessary to realize the Poincaré algebra. In particular, the Poisson brackets should be redefined to get reasonable results.

The Hamiltonian formalism proposed by Ashtekar,[1] which uses as field variables spinor soldering forms or triads instead of a 3-metric, is of considerable interest for the quantum theory of gravity. In Refs. 2,3 these applications have already been developed. However, until now not enough attention has been paid to quantization of the asymptotically flat spacetime . In such a problem an infinite group of diffeomorphisms which preserve boundary conditions at spatial infinity is of great importance. The Poincaré group can be factored out from it. In the ordinary canonical formalism, the Poincaré group generators are expressed in terms of surface integrals taken over a two-dimensional sphere at spatial infinity, as was shown for the first time by Regge and Teitelboim[4].

The two-fold nature of these surface integrals was disclosed in Ref. 4: they guarantee differentiability of the Hamiltonian at the prescribed boundary conditions and they also enter the Poisson brackets algebra

$$\{H(N^\alpha), H(M^\beta)\} \approx H(L^\gamma), \tag{1}$$

where

$$L^\gamma = [N, M]^\gamma, \tag{2}$$

and every vector field represents coordinate transformation from the asymptotic Poincaré group. The second aspect was discussed much less than the first one. But the straightforward calculation of all the surface terms in the Poisson brackets, in general case first done in Ref. 5, gives us many advantages, for example, a possibility to search for new boundary conditions or to evaluate central charges. Later this method was also mentioned in Ref. 6, with a remark that "such a calculation is typically very cumbersome". The same authors have proved in other publication[7] that a Poisson bracket of two differentiable in Regge-Teitelboim's sense generators is also a differentiable generator.

Here we will show how the Poincaré group manifests itself in Ashtekar's canonical formalism where the boundary conditions for new variables are taken from Refs. 8,9. As the basic variable we choose the triad related to the 3-metric as follows:

$$\gamma\gamma^{ij} = \tilde{E}^{ia}\tilde{E}^{ja}, \tag{3}$$

(in our notations, the triad indices are: $a, b, c, ...$; the spatial ones are: $i, j, k, ...$). The Hamiltonian contains three additional constraints as compared to the standard General Relativity:

$$H = \int [\underset{\sim}{N}\tilde{E}^{ia}\tilde{E}^{jb}F^c_{ij}\epsilon^{abc} + 2iN^i\tilde{E}^{ja}F^a_{ij} + \hat{N}^a\mathcal{D}_i\tilde{E}^{ia}]d^3x. \tag{4}$$

These new constraints generate the triad rotations and are analogous to the Gaussian constraints of the gauge theories. Conjugate to $\tilde{E}^{ia}$ is the connection A^a_i:

$$\{\tilde{E}^{ia}(x), A^b_j(y)\} = \frac{i}{2}\delta^i_j\delta^{ab}\delta(x,y). \tag{5}$$

The constraint algebra was obtained in Refs. 8,9, where, as usual, the surface integrals arising in calculations of the Poisson brackets by the formula

$$\{H_1, H_2\} = \frac{i}{2}\int\left[\frac{\delta H_1}{\delta\tilde{E}^{ia}}\frac{\delta H_2}{\delta A^a_i} - (1\leftrightarrow 2)\right]d^3x, \tag{6}$$

were not taken into account.

Following our approach proposed in Ref. 5, we find the Poisson brackets between the constraints not discarding any surface terms. The appearance of these surface integrals is not very transparent[11] and we do not display them here. According to Regge-Teitelboim's ideology and to Brown-Henneaux's theorem[7] we can expect these surface terms to coincide with those obtained through the Hamiltonian variation at Ashtekar's boundary conditions[8,9]:

$$H(N^\alpha) \approx \oint 2\underset{\sim}{N}\epsilon^{abc}\tilde{E}^{ia}\tilde{E}^{jb}A^c_j dS_i + 2i\oint(N^i\tilde{E}^{ja} - N^j\tilde{E}^{ia})A^a_j dS_i, \tag{7}$$

but unfortunately the results are different. For example, for the exact Schwarzschild solution,[10]

$$\tilde{E}^{ia} = (1-k/r)^{-1/2}\delta_{ia} + [1-(1-k/r)^{-1/2}]n_i n_a, \tag{8}$$

$$A^a_i = 1/r[(1-k/r)^{1/2} - 1]\epsilon_{iab}n_b, \tag{9}$$

where $n_i = x^i/r$, by commutating the spatial translation $N^i = \lambda^i$ and the rotation $M^j = \bar{\omega}_{jk}x^k$, instead of zero momentum we have

$$\{H(N^i), H(M^j)\} = i\frac{4\pi k}{3}\epsilon_{ijk}\lambda_i\bar{\omega}_{jk}, \tag{10}$$

and by commutating the same spatial translation and the boost $\underset{\sim}{M} = \bar{B}_k x^k$, instead of the Schwarzschild mass $8\pi k$ multiplied by time translation $\lambda^i\bar{B}_i$ we have

$$\{H(N^i), H(\underset{\sim}{M})\} = \frac{8\pi k}{3}\lambda^i\bar{B}_i. \tag{11}$$

The reason for these unpleasant results as we have found in Ref. 11 is that the Ashtekar's transformation[1,8,9] is not strictly canonical. When surface integrals are taken into account the Poisson bracket

$$\{ \int N^{ia}(x) A_i^a(x) d^3x, \int M^{jb}(y) A_j^b(y) d^3x \} \tag{12}$$

is not zero and the correct Poisson bracket for Hamiltonian generators should be

$$\{H_1, H_2\} = \{H_1, H_2\}_{old} + \{H_1, H_2\}_{correction}, \tag{13}$$

where

$$\{H_1, H_2\}_{correction} = \int d^3x \int d^3y \frac{\delta H_1}{\delta A_i^a(x)} \frac{\delta H_2}{\delta A_j^b(y)} \{ A_i^a(x), A_j^b(y)\} =$$

$$= \frac{i}{4} \oint \frac{dS_k}{E} \epsilon^{acd} (\delta_j^k \delta^{bd} E_{ic} - E_{ib} E_{jc} E^{kd}) \left[\frac{\delta H_1}{\delta A_i^a} \frac{\delta H_2}{\delta A_j^b} - (1 \leftrightarrow 2) \right]. \tag{14}$$

With this redefinition of the Poisson brackets the algebra of Poincaré group generators can be definitely realized for the Schwarzschild solution and the generators have values that coincide with the Ashtekar's expressions.

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