Canonical formalism of bigravity

Vladimir O. Soloviev

International University "Dubna", Institute for High Energy Physics (Protvino)

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Dark energy is a problem

Contents

On massive gravity, old and new

- Boulware-Deser ghost (negative kinetic energy)
- Causality problems (two different light cones)
- Nonperturbative solutions for small distances (Vainshtein radius)
- No Hamiltonian constraint (arbitrary initial conditions)

...“*the subject remained moribund until the recent (independent) rediscovery (dRGT, 2011) of the results by Wess-Zumino (1970). This exhumation has, unsurprisingly, generated an immense industry. Our purpose is to re-inter at least one model*”... *S. Deser, A. Waldron (arxiv:1212.5835)*

History and terminology

- Bimetric theory: *N. Rosen (1940)*
- f-g gravity: *Isham, Salam, Strathdee (1971)*
- Strong gravitation: *Zumino (1971)*
- Bigravity: *Damour, Kogan (2002)*
- Bimetric gravity: *Hassan, R. Rosen (2011)*

Bigravity

The bigravity Lagrangian

$$\mathcal{L} = \mathcal{L}^{(f)} + \mathcal{L}^{(g)} - \sqrt{-f}\, U(f_{\mu\nu}, g_{\mu\nu}).$$

is composed of two almost independent parts

$$\mathcal{L}^{(f)} = \frac{1}{16\pi G^{(f)}} \sqrt{-f} f^{\mu\nu} R^{(f)}_{\mu\nu} + \mathcal{L}^{(f)}_M(\psi^A, f_{\mu\nu}),$$

$$\mathcal{L}^{(g)} = \frac{1}{16\pi G^{(g)}} \sqrt{-g} g^{\mu\nu} R^{(g)}_{\mu\nu} + \mathcal{L}^{(g)}_M(\phi^A, g_{\mu\nu}),$$

and the potential of their interaction

$$\sqrt{-f}\, U(f_{\mu\nu}, g_{\mu\nu}) \quad \text{or} \quad \sqrt[4]{fg}\, V(f_{\mu\nu}, g_{\mu\nu}) \quad \text{or} \quad \sqrt{-g}\, W(f_{\mu\nu}, g_{\mu\nu}).$$

Degrees of freedom

	GR	biG (general)	biG (dRGT)	PM biG ?
variables ($2N$)	(γ_{ij}, π^{ij})	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij})	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij})	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij})
1st class (N_1)	$\mathcal{H}, \mathcal{H}_i$	$\mathcal{R}, \mathcal{R}_i$	$\mathcal{R}, \mathcal{R}_i$	$\mathcal{R}, \mathcal{R}_i$ $\mathcal{S}, \Omega$
2nd class (N_2)	—	—	$\mathcal{S}, \Omega$	—
DoF	2	8	7	6

$$\text{DoF} = \frac{2N - 2N_1 - N_2}{2}$$

	biG (general)	the same	biG (dRGT)	the same
variables ($2N$)	(γ_{ij},π^{ij}) (η_{ij},Π^{ij}) 24	(γ_{ij},π^{ij}) (η_{ij},Π^{ij}) (u^a,π_a) 32	(γ_{ij},π^{ij}) (η_{ij},Π^{ij}) 24	(γ_{ij},π^{ij}) (η_{ij},Π^{ij}) (u^a,π_a) 32
1st class (N_1)	$\mathcal{R},\mathcal{R}_i$ 4	$\mathcal{R},\mathcal{R}_i$ 4	$\mathcal{R},\mathcal{R}_i$ 4	$\mathcal{R},\mathcal{R}_i$ 4
2nd class (N_2)	0	$\pi_a,\mathcal{S}_a$ 8	$\mathcal{S},\Omega$ 2	$\pi_i,\mathcal{S}_i$ $\pi,\mathcal{S},\Omega,\Psi$ 10
DoF	8	8	7	7

$$\mathrm{DoF} = \frac{2N - 2N_1 - N_2}{2}$$

Results: axioms for the potential

1. We have a differentiable function $\tilde{U} = \tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij})$.
2. Diffeomorphism invariance requires

$$\begin{aligned} 2\eta_{ik}\frac{\partial \tilde{U}}{\partial \eta_{jk}} + 2\gamma_{ik}\frac{\partial \tilde{U}}{\partial \gamma_{jk}} - u^j \frac{\partial \tilde{U}}{\partial u^i} - \delta^j_i \tilde{U} &= 0, \\ 2u^j \gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ik}} - u^i u \frac{\partial \tilde{U}}{\partial u} + \left(\eta^{ik} - u^2\gamma^{ik} - u^i u^k\right)\frac{\partial \tilde{U}}{\partial u^k} &= 0. \end{aligned}$$

3. The big Hessian matrix is to be degenerate

$$\left|\frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}\right| = 0, \qquad u^a = (u, u^i).$$

4. The small Hessian matrix is to be nondegenerate

$$\left|\frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j}\right| \neq 0, \qquad i = 1, 2, 3.$$

Our publications

Bigravity in Kuchar's Hamiltonian formalism. 1. The general case
Vladimir O. Soloviev and Margarita V. Tchichikina
arXiv:1211.6530;

Bigravity in Kuchar's Hamiltonian formalism. 2. The special case
Vladimir O. Soloviev and Margarita V. Tchichikina
arXiv:1302.5096 (2nd version - April 2013)

de Rham, Gabadadze, Tolley

Both the modern massive gravity and the bigravity (sometimes called as the bimetric gravity) exploit the dRGT potential expressed by means of the matrix square root. Let

$$Y^{\alpha}_{\beta} = g^{\alpha\mu} f_{\mu\beta},$$

then

$$X = \sqrt{Y}, \qquad Y^{\alpha}_{\beta} = X^{\alpha}_{\mu} X^{\mu}_{\beta}.$$

The potential is formed as a linear combination of coefficients of the characteristic polynomial of matrix X^{α}_{β}

$$P(\lambda) = Det(X - \lambda I),$$

i.e. as a linear combination of symmetric polynomials of this matrix eigenvalues

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_2\lambda_4 + \lambda_1\lambda_3,$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2,$$

$$\lambda_1\lambda_2\lambda_3\lambda_4,$$

or by means of the traces of the matrix polynomials,

$$SpX,$$

$$\frac{1}{2}\left((SpX)^2 - SpX^2\right),$$

$$\frac{1}{6}\left((SpX)^3 - 3SpXSpX^2 + 2SpX^3\right),$$

$$\frac{1}{24}\left((SpX)^4 - 6(SpX)^2SpX^2 + 3(SpX^2)^2 + 8SpXSpX^3 - 6SpX^4\right).$$

The time

- The time as a coordinate

$$\dot{A} = \frac{\partial A}{\partial x^0}$$

- The time as a new parameter

$$\dot{A} = \frac{\partial A}{\partial \tau}$$

- The time as a vector field

$$\dot{A} = N^\alpha \nabla_\alpha A$$

Kuchař's approach

- The second coordinate frame for space-time is introduced τ, x^i, correlated with the choice of the family of hypersurfaces $\tau = \mathrm{const}$.
- One-to-one differentiable map is supposed to take place

$$X^\alpha = e^\alpha(\tau, x^i),$$

where $e^\alpha(\tau, x^i)$ are called embedding functions.

- The local basis $\{n^\alpha, e_i^\alpha\}$ with the unit normal $n_\alpha e_i^\alpha = 0$, $n_\alpha n^\alpha = -1$ is constructed.
- All the vectors and tensors are expanded over this basis:

$$A^\alpha = A^\perp n^\alpha + A^i e_i^\alpha,$$

$$A^\perp = -A^\alpha n_\alpha, \qquad A^i = A^\alpha e^i_\alpha.$$

- In the process of this expansion of the covariant derivatives

$$A_{i;j} = A_{i|j} - A_{\perp} K_{ij}, \qquad A_{\perp;i} = A_{\perp|i} - A^j K_{ij},$$

  the external curvature tensor of the hypersurface appears

$$K_{ij} = -n_{\alpha;\beta} e^{\alpha}_i e^{\beta}_j .$$

- Time derivative is defined as follows

$$\dot{A}_{\alpha} = N^{\beta} A_{\alpha;\beta}, \qquad N^{\alpha} = \frac{\partial e^{\alpha}}{\partial \tau} = N n^{\alpha} + N^i e^{\alpha}_i ,$$

  for example, for the induced metric $\gamma_{ij} = g_{\alpha\beta} e^{\alpha}_i e^{\beta}_j$ we obtain

$$\dot{\gamma}_{ij} = N_{i|j} + N_{j|i} - 2NK_{ij}.$$

- The space-time metric decomposition is as follows
  $g^{\mu\nu} = -n^{\mu} n^{\nu} + e^{\mu}_i e^{\nu}_j \gamma^{ij}$.

An example: the scalar field

The action is

$$S = \int d^4 X \mathcal{L} = -\int d^4 X \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + U(\phi) \right)$$

The transform to $3+1$-notations:

$$\mathcal{L} = \frac{1}{2} N \sqrt{\gamma} \left(\phi_{,\perp} \phi_{,\perp} - \gamma^{ij} \phi_{,i} \phi_{,j} - 2U(\phi) \right),$$

where

$$\dot{\phi} = N^\alpha \phi_{,\alpha} = -N \phi_{,\perp} + N^i \phi_{,i}.$$

The momentum is defined in a standard way

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\sqrt{\gamma} \phi_{,\perp} = \frac{\sqrt{\gamma}}{N} \left(\dot{\phi} - N^i \phi_{,i} \right),$$

and the velocity can be expressed as a function of the momentum as follows

$$\dot{\phi} = \frac{N}{\sqrt{\gamma}}\pi_\phi + N^i\phi_{,i}.$$

After making the Legandre transform the scalar field action takes a form

$$S = \int d\tau d^3x \left(\pi_\phi\dot{\phi} - N\mathcal{H} - N^i\mathcal{H}_i\right)$$

where

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left(\frac{\pi_\phi^2}{2} + \frac{1}{2}\gamma\gamma^{ij}\partial_i\phi\partial_j\phi + \gamma U(\phi)\right), \quad \mathcal{H}_i = \pi_\phi\phi_{,i}.$$

"Quantization on curved hypersurfaces"

If, following Dirac, we will give to the embedding functions a new status, and consider them as dynamical variables, then we introduce the conjugate momenta

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{e}^\alpha} = \frac{\partial \mathcal{L}}{\partial N^\alpha}$$

and arrive at the first class constraints

$$\mathcal{R} \equiv p + \mathcal{H} = 0, \qquad \mathcal{R}_i \equiv p_i + \mathcal{H}_i = 0,$$

satisfying the algebra

$$\begin{aligned} \{\mathcal{R}_i(x), \mathcal{R}_j(y)\} &= \mathcal{R}_i(y)\delta_{,j}(x-y) + \mathcal{R}_j(x)\delta_{,i}(x-y), \\ \{\mathcal{R}_i(x), \mathcal{R}(y)\} &= \mathcal{R}(x)\delta_{,i}(x-y), \\ \{\mathcal{R}(x), \mathcal{R}(y)\} &= \left[\gamma^{ij}(x)\mathcal{R}_j(x) + \gamma^{ij}(y)\mathcal{R}_j(y)\right]\delta_{,i}(x-y). \end{aligned}$$

The Hamiltonian contains arbitrary functions and equals a linear combination of the constraints

$$H = \int d^3x \left(\lambda \mathcal{R} + \lambda^i \mathcal{R}_i \right) .$$

The evolution of embedding variables is determined by Lagrangian multipliers λ^A, and by their initial data: $e^\alpha(0, x^i)$.

In particular, we can take a control in our own hands and deform the hypersurface according to our taste.

The evolution parameter can be taken other than time.

General Relativity

$$S=\int d^4X\sqrt{-g}g^{\mu\nu}R_{\mu\nu}.$$

Again let us use a second coordinate frame τ, x^i with the embedding functions $X^\alpha = e^\alpha(\tau, x^i)$. The space-time metric $g_{\mu\nu}$ in the procedure of $3+1$-decomposition is an instrument to construct both the basis $\{n^\alpha, e_i^\alpha\}$, and the dynamical variables as projections that are to be calculated with this basis. Now four of these projections occur trivial

$$g_{\perp\perp}=-1, \qquad g_{\perp i}=0,$$

so there are only 6 dynamical variables $\gamma_{ij}=g_{\alpha\beta}e_i^\alpha e_j^\beta$.

The Riemann and Ricci tensors could be expressed by means of the external curvature of the hypersurface

$$K_{ij} = \frac{1}{2N} \left(N_{i|j} + N_{j|i} - \dot{\gamma}_{ij} \right),$$

and lapse-shift functions N, N^i with their derivatives.
As a result, the Lagrangian density appears as follows

$$N\sqrt{\gamma} \left(\tilde{R} - K^2 + \mathrm{Sp} K^2 \right),$$

where $\tilde{R}$ is a scalar curvature of the hypersurface determined exclusively by the induced metric γ_{ij}.

We ignore all the surface terms.

We conclude that the Lagrangian contains velocities for induced metric only and not for $N^A = (N, N^i)$, so there are primary constraints

$$\pi_A = 0.$$

We have canonical Poisson brackets

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \gamma_{ij}} \frac{\delta G}{\delta \pi^{ij}} + \frac{\delta F}{\delta N^A} \frac{\delta G}{\delta \pi_A} - (F \leftrightarrow G) \right) d^3x,$$

and the following Hamiltonian

$$H = \int \left(N\mathcal{H} + N^i \mathcal{H}_i + \lambda^A \pi_A \right) d^3x.$$

Preservation of the primary constraints in the course of evolution requires

$$\dot{\pi}_A = \{\pi_A, H\} = 0,$$

so we obtain secondary constraints

$$\mathcal{H} = -\sqrt{\gamma}R^{(3)} + \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \mathrm{Sp}\pi^2\right) = 0, \qquad \mathcal{H}_i = -2\pi^j_{i|j} = 0,$$

and they are in involution, what is more, their algebra coincides with the algebra of constraints in the parametrized field theories

$$\begin{aligned}
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_i(y)\delta_{,j}(x-y) + \mathcal{H}_j(x)\delta_{,i}(x-y), \\
\{\mathcal{H}_i(x), \mathcal{H}(y)\} &= \mathcal{H}(x)\delta_{,i}(x-y), \\
\{\mathcal{H}(x), \mathcal{H}(y)\} &= \left[\gamma^{ij}(x)\mathcal{H}_j(x) + \gamma^{ij}(y)\mathcal{H}_j(y)\right]\delta_{,i}(x-y).
\end{aligned}$$

Should we treat N, N^i as canonical variables or as Lagrangian multipliers? The last is equivalent to taking gauge conditions and moving to trivial Dirac brackets.

$$\{F, G\}_D = \int \left(\frac{\delta F}{\delta \gamma_{ij}} \frac{\delta G}{\delta \pi^{ij}} - \frac{\delta G}{\delta \gamma_{ij}} \frac{\delta F}{\delta \pi^{ij}} \right) d^3x,$$

DoF=(the number of canonical variables - the number of first class constraints $\times 2)/2 = (12 - 4 * 2)/2 = 2$.

γ_{ij}, π^{ij} — canonical variables, $\mathcal{H}, \mathcal{H}_i$ — first class constraints, N, N^i — Lagrangian multipliers,

$$\mathrm{H} = \int (N\mathcal{H} + N^i \mathcal{H}_i) d^3x.$$

Notations for bigravity

One-parametrical family of spacelike (in both metrics $f_{\mu\nu}, g_{\mu\nu}$) hypersurfaces is defined by equations

$$X^\alpha = e^\alpha(\tau, x^i),$$

then

$$N^\alpha \equiv \frac{\partial e^\alpha}{\partial \tau}, \qquad e_i^\alpha \equiv \frac{\partial e^\alpha}{\partial x^i}.$$

The induced on hypersurfaces metrics are

$$\gamma_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu, \quad \eta_{ij} = f_{\mu\nu} e_i^\mu e_j^\nu.$$

There are two different fields of unit normals for any hypersurface, $n^\alpha(x^i)$ and $\bar{n}^\alpha(x^i)$ related by equation

$$n^\alpha = u\bar{n}^\alpha + u^i e_i^\alpha,$$

and there are two bases (n^α, e_i^α), $(\bar{n}^\alpha, e_i^\alpha)$ applicable for $3+1$-decompositions of the spacetime vectors and tensors.

The suitable variables are the following

$$u = \frac{1}{\sqrt{-g^{\perp\perp}}}, \qquad u^i = -\frac{g^{\perp i}}{g^{\perp\perp}}.$$

They have a transparent geometric meaning: u is the inverse of a norm (calculated in g-metric) of unit normal vector n^α, constructed according to f-metric, and u^i are projections (in g-metric) of the hypersurface coordinate basis vectors onto n^α:

$$u = \frac{1}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}, \qquad u^i = \frac{g^{\mu\nu} n_\mu e^i_\nu}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}.$$

Two sets of lapses and shifts are related by formulas:

$$\bar{N} = uN, \qquad \bar{N}^i = N^i + u^i N,$$

inverse formulas are

$$u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}.$$

In the standard ADM-decomposition (coordinate basis):

$$||g^{\mu\nu}|| = \begin{pmatrix} -N^{-2} & N^j N^{-2} \\ N^i N^{-2} & \gamma^{ij} - N^i N^j N^{-2} \end{pmatrix},$$

In Kuchař's decomposition (f-metric basis):

$$||g^{\mu\nu}|| = \begin{pmatrix} -u^{-2}[n^\mu n^\nu] & u^j u^{-2}[n^\mu e^\nu_j] \\ u^i u^{-2}[e^\mu_i n^\nu] & (\gamma^{ij} - u^i u^j u^{-2})[e^\mu_i e^\nu_j] \end{pmatrix},$$

Bigravity Hamiltonian

Let

$$\tilde{U} = \sqrt{\eta} U.$$

$$\begin{aligned} \mathrm{H}_{\mathrm{canonical}} &= \mathrm{H}^{(f)} + \mathrm{H}^{(g)} + \int d^3x N \sqrt{\eta} U \\ &= \int d^3x \left(N\mathcal{H} + N^i \mathcal{H}_i + \bar{N}\bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i + N\tilde{U} \right), \end{aligned}$$

Another form is as follows

$$\mathrm{H}_{\mathrm{canonical}} = \int d^3x \left[N \left(\mathcal{H} + u\bar{\mathcal{H}} + u^i \bar{\mathcal{H}}_i + \tilde{U} \right) + N^i \left(\mathcal{H}_i + \bar{\mathcal{H}}_i \right) \right].$$

Primary constraints

$$\pi_N = 0, \quad \pi_{N^i} = 0, \quad \pi_u = 0, \quad \pi_{u^i} = 0.$$

Secondary constraints

$$\mathcal{R} \equiv \mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \tilde{U} = 0,$$

$$\mathcal{R}_i \equiv \mathcal{H}_i + \bar{\mathcal{H}}_i = 0,$$

$$\mathcal{S} \equiv \bar{\mathcal{H}} + \frac{\partial \tilde{U}}{\partial u} = 0, \qquad \mathcal{S}_i \equiv \bar{\mathcal{H}}_i + \frac{\partial \tilde{U}}{\partial u^i} = 0.$$

The Jacobian for $\mathcal{S}, \mathcal{S}_i$ is the Hessian for $\tilde{U}$:

$$\frac{D(\mathcal{S}, \mathcal{S}_i)}{D(u, u^j)} = \left| \frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b} \right|.$$

Algebra of first class constraints

$$
\begin{aligned}
\{\mathcal{R}_x, \mathcal{R}_y\} &= \left(\eta^{ij}\mathcal{R}_j + uu^i\mathcal{S} - (\eta^{ij} - u^2\gamma^{ij} - u^i u^j)\mathcal{S}_j + Q^i\right)_x \delta_{,i}(x,y) \\
&- \left(\eta^{ij}\mathcal{R}_j + uu^i\mathcal{S} - \left(\eta^{ij} - u^2\gamma^{ij} - u^i u^j\right)\mathcal{S}_j + Q^i\right)_y \delta_{,i}(y,x) \\
&\approx \left(\eta^{ij}\mathcal{R}_j + Q^i\right)(x)\delta_{,i}(x,y) - \left(\eta^{ij}\mathcal{R}_j + Q^i\right)(y)\delta_{,i}(y,x),
\end{aligned}
$$

$$
\begin{aligned}
\{\mathcal{R}_{ix}, \mathcal{R}_y\} &= \mathcal{R}(x)\delta_{,i}(x,y) + u_{,i}\mathcal{S}\delta(x,y) + \\
&+ \frac{\partial}{\partial x^j}\left(Q^j_i(x)\delta(x,y)\right) + \\
&+ \frac{\partial}{\partial x^j}\left(u^j\mathcal{S}_i(x)\delta(x,y)\right) + u^j_{,i}\mathcal{S}_j\delta(x.y) \approx \\
&\approx \mathcal{R}(x)\delta_{,i}(x,y) + \frac{\partial}{\partial x^j}\left(Q^j_i(x)\delta(x,y)\right),
\end{aligned}
$$

$$
\{\mathcal{R}_i(x), \mathcal{R}_j(y)\} = \mathcal{R}_i(y)\delta_{,j}(x,y) + \mathcal{R}_j(x)\delta_{,i}(x,y).
$$

Requirements for the potential

$$Q^i_k = 2\eta_{jk}\frac{\partial \tilde{U}}{\partial \eta_{ij}} + 2\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ij}} - u^i\frac{\partial \tilde{U}}{\partial u^k} = \delta^i_k \tilde{U},$$

$$Q^\ell = 2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{k\ell}} - u^\ell u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{k\ell} - u^2\gamma^{k\ell} - u^k u^\ell\right)\frac{\partial \tilde{U}}{\partial u^k} = 0.$$

Second class constraints (the general case)

Let $u^a = (u, u^i)$, $\pi_a = (\pi_u, \pi_{u^i})$, $\mathcal{S}_a = (\mathcal{S}, \mathcal{S}_i)$, $a = 1, .., 4$. χ_A, $A = 1, .., 8$, then

$$\chi_A = (\pi_a, \mathcal{S}_a)\,,$$

$$||\{\chi_A(x), \chi_B(y)\}|| = \begin{pmatrix} \mathbf{0} & -\mathbf{L}(x)\delta(x,y) \\ \mathbf{L}(x)\delta(x,y) & \mathbf{K}(x,y) \end{pmatrix},$$

where

$$\mathbf{L}_{ab}(x) = \frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}(x),$$

$$\mathbf{K}_{ab}(x,y) = \{\mathcal{S}_a(x), \mathcal{S}_b(y)\}\,.$$

As Jacobian for $\mathcal{S}, \mathcal{S}_i$ is the Hessian for $\tilde{U}$ in case it is non-degenerate, it is possible to solve $\mathcal{S}, \mathcal{S}_i$ for u, u^i variables and exclude all the 2nd class constraints $(\pi_a, \mathcal{S}_b)$. We can also use Dirac brackets. Then we have DoF=8.

Therefore it is necessary to require a zero Jacobian, i.e. zero Hessian for a theory with less DoF. This condition on the bigravity potential is known as the Monge-Ampère equation

$$\left| \frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b} \right| = 0.$$

Second class constraints (the exceptional case)

Let we consider less second class constraints, denote them as follows $\tilde{\chi}_A$, $A = 1, .., 6$, where

$$\tilde{\chi}_A = (\pi_i, \mathcal{S}_i)\,,$$

then let their Poisson brackets form a non-degenerate matrix

$$||\{\tilde{\chi}_A(x), \tilde{\chi}_B(y)\}|| = \begin{pmatrix} \mathbf{0} & -\tilde{\mathbf{L}}(x)\delta(x,y) \\ \tilde{\mathbf{L}}(x)\delta(x,y) & \tilde{\mathbf{K}}(x,y) \end{pmatrix},$$

where

$$\tilde{\mathbf{L}}(x) = ||L_{ij}(x)|| = \frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j}(x), \qquad \tilde{\mathbf{K}}(x,y) = ||K_{ij}(x,y)||$$

$$K_{ij}(x,y) = \{\mathcal{S}_i(x), \mathcal{S}_j(y)\}\,.$$

Dirac brackets

If matrix $\widetilde{\mathbf{L}}$ is non-degenerate, then matrix $||\{\tilde{\chi}_A, \tilde{\chi}_B\}||$ is invertible:

$$\widetilde{\mathbf{C}}(x,y) = \begin{pmatrix} \widetilde{\mathbf{L}}^{-1}(x)\widetilde{\mathbf{K}}(x,y)\widetilde{\mathbf{L}}^{-1}(y) & \widetilde{\mathbf{L}}^{-1}(x)\delta(x,y) \\ -\widetilde{\mathbf{L}}^{-1}(x)\delta(x,y) & \mathbf{0} \end{pmatrix},$$

and we can construct Dirac brackets:

$$\{F,G\}_{\tilde{D}} = \{F,G\} - \int dx \int dy \{F, \tilde{\chi}_A(x)\}\widetilde{\mathbf{C}}^{AB}(x,y)\{\tilde{\chi}_B(y), G\}.$$

For the general potential the Dirac bracket of constraint $\mathcal{S}$ with itself is nonzero:

$$\{\mathcal{S}_x, \mathcal{S}_y\}_D = (\Theta^i - \bar{U}^i \mathcal{S})_x \delta_{,i}(x,y) - (\Theta^i - \bar{U}^i \mathcal{S})_y \delta_{,i}(y,x),$$

where

$$\Theta^i = \left(\bar{U}^k \hat{D} \left(\delta^i_k - 2\gamma_{jk} \frac{\partial}{\partial \gamma_{ij}} \right) - \gamma^{ij} \frac{\partial}{\partial u^j} \right) \tilde{U},$$

and

$$\hat{D} = \frac{\partial}{\partial u} - \bar{U}^i \frac{\partial}{\partial u^i}.$$

The Dirac bracket between constraints $\mathcal{S}$ and $\mathcal{R}$ is, in general, not weakly equal to zero:

$$\{\mathcal{R}(x), \mathcal{S}(y)\}_{\tilde{D}} = \left(u^i - u\bar{U}^i\right)\mathcal{S}(x)\delta_{,i}(x,y) - \left(u(\bar{U}^i\mathcal{S})_{,i} + \Omega\right)\delta(x,y),$$

and so, provides us with a new (tertiary) constraint $\Omega = 0$. It is possible also to calculate the following brackets

$$\begin{aligned} \{\mathcal{R}(x), \pi_u(y)\}_{\tilde{D}} &= \mathcal{S}(x)\delta(x,y), \\ \{\Omega(x), \pi_u(y)\}_{\tilde{D}} &= \Theta^i(x)\delta_{,i}(x,y) - \Theta^i(y)\delta_{,i}(y,x). \end{aligned} \tag{1}$$

On the base of implicit solution of the Monge-Ampére equation found by D. Fairlie and A. Leznov (1994) it is possible to prove that $\Theta^i = 0$. The two requirements might be fulfilled at once by this:

1. the tertiary constraint Ω appears;
2. this constraint will not depend on variable u.

$(1, -\bar{U}^i)$ is a null-vector of the Hessian matrix, so, we let use a notation from Fairlie-Leznov $\xi^i = -\bar{U}^i$. It is suitable to replace potential $\tilde{U}$ by function $V(\xi^i, \eta_{ij}, \gamma_{ij})$, such that

$$\frac{\partial \tilde{U}}{\partial u^i} = \frac{\partial V}{\partial \xi^i}, \qquad \frac{\partial \tilde{U}}{\partial u} = V - \xi^i \frac{\partial V}{\partial \xi^i},$$

then it follows that

$$\hat{D}\tilde{U} = \left(\frac{\partial}{\partial u} - \bar{U}^k \frac{\partial}{\partial u^k} \right) \tilde{U} = \left(\frac{\partial}{\partial u} + \xi^k \frac{\partial}{\partial u^k} \right) \tilde{U} = V. \qquad (2)$$

Integrability condition for Monge-Ampére equation is as follows

$$\hat{D}\xi^i = \frac{\partial \xi^i}{\partial u} + \xi^k \frac{\partial \xi^i}{\partial u^k} = 0.$$

At last, it is proved that any implicit solution for the above system of equations can be written as

$$u^i - u\xi^i = (V^{-1})^{ik}\frac{\partial W}{\partial \xi^k},$$

where $W = W(\xi^i, \eta_{ij}, \gamma_{ij})$ is an arbitrary function, and

$$V_{ik} = \frac{\partial^2 V}{\partial \xi^i \partial \xi^k}.$$

If we apply twice differential operator $\hat{D}$ to the equation found above

$$Q^\ell = 2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{k\ell}} - u^\ell u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{k\ell} - u^2\gamma^{k\ell} - u^k u^\ell\right)\frac{\partial \tilde{U}}{\partial u^k} = 0.$$

then we prove that

$$\Theta^i = 0.$$

Quaternary constraint Ψ appears from

$$\{\Omega_x, \mathrm{H}\}_D \approx \int d^3y \{\Omega_x, \mathcal{R}_y\}_D N_y \approx \int d^3y \Psi_x \delta(x,y) N_y = 0,$$

Ψ is linear in variable u, as $\mathcal{R} = u\mathcal{S} + \dots$ and so can be solved for it. The Jacobi identity

$$\{\Omega, \{\pi_u, \mathcal{R}\}_{\tilde{D}}\}_{\tilde{D}} + \{\mathcal{R}, \{\Omega, \pi_u\}_{\tilde{D}}\}_{\tilde{D}} + \{\pi_u, \{\mathcal{R}, \Omega\}_{\tilde{D}}\}_{\tilde{D}} = 0,$$

gives us the following result

$$\{\pi_u(y), \Psi(x)\}_{\tilde{D}} \delta(x,z) = -\{\Omega(x), \mathcal{S}(z)\}_{\tilde{D}} \delta(z,y),$$

so if $\{\Omega, \mathcal{S}\}_{\tilde{D}}$ is nonzero, then $\{\pi_u, \Psi\}_{\tilde{D}}$ is nonzero also. But it is easy to check that there are nonzero terms in $\{\Omega, \mathcal{S}\}_{\tilde{D}}$, for example,

$$[[V, \mathcal{H}], \bar{\mathcal{H}}] = \frac{4\kappa^{(f)}\kappa^{(g)}}{\sqrt{\eta\gamma}} \left(\Pi_{ij} - \eta_{ij}\frac{\Pi}{2}\right) \frac{\partial^2 V}{\partial\eta_{ij}\partial\gamma_{mn}} \left(\pi_{mn} - \gamma_{mn}\frac{\pi}{2}\right) \neq 0,$$

that cannot be canceled by other terms, and does not appear in other constraints. Therefore both Dirac brackets above are nonzero.

Table of Dirac brackets between constraints

$\{,\}_{\tilde{D}}$	$\pi_u(y)$	$\Psi(y)$	$\Omega(y)$	$\mathcal{S}(y)$	$\mathcal{R}(y)$	$\mathcal{R}_j(y)$
$\pi_u(x)$ (primary)	0	$\neq 0$	$-\hat{\Theta} = 0$	0	≈ 0	0
$\Psi(x)$ (quaternary)	$\neq 0$					
$\Omega(x)$ (tertiary)	$\hat{\Theta} = 0$			$\neq 0$	$\Psi \approx 0$	≈ 0
$\mathcal{S}(x)(secondary)$	0		$\neq 0$	$\hat{\Theta} = 0$	≈ 0	≈ 0
$\mathcal{R}(x)$ (secondary)	≈ 0		$-\Psi \approx 0$	≈ 0	≈ 0	≈ 0
$\mathcal{R}_i(x)$ (secondary)	0		≈ 0	≈ 0	≈ 0	≈ 0

	biG (general)	the same	biG (dRGT)	the same
variables ($2N$)	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij}) 24	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij}) (u^a, π_a) 32	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij}) 24	(γ_{ij}, π^{ij}) (η_{ij}, Π^{ij}) (u^a, π_a) 32
1st class (N_1)	$\mathcal{R}, \mathcal{R}_i$ 4	$\mathcal{R}, \mathcal{R}_i$ 4	$\mathcal{R}, \mathcal{R}_i$ 4	$\mathcal{R}, \mathcal{R}_i$ 4
2nd class (N_2)	 0	$\pi_a, \mathcal{S}_a$ 8	 $\mathcal{S}, \Omega$ 2	$\pi_i, \mathcal{S}_i$ $\pi, \mathcal{S}, \Omega, \Psi$ 10
DoF	8	8	7	7

$$\text{DoF} = \frac{2N - 2N_1 - N_2}{2}$$

Results: axioms for the potential

1. We have a differentiable function $\tilde{U} = \tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij})$.
2. Diffeomorphism invariance requires

$$
\begin{aligned}
2\eta_{ik}\frac{\partial \tilde{U}}{\partial \eta_{jk}} + 2\gamma_{ik}\frac{\partial \tilde{U}}{\partial \gamma_{jk}} - u^j\frac{\partial \tilde{U}}{\partial u^i} - \delta^j_i \tilde{U} &= 0, \\
2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ik}} - u^i u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{ik} - u^2\gamma^{ik} - u^i u^k\right)\frac{\partial \tilde{U}}{\partial u^k} &= 0.
\end{aligned}
$$

3. The big Hessian matrix is to be degenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}\right| = 0, \qquad u^a = (u, u^i).
$$

4. The small Hessian matrix is to be nondegenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j}\right| \neq 0, \qquad i = 1, 2, 3.
$$

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