Bigravity and massive gravity in Hamiltonian approach

Vladimir O. Soloviev

Institute for High Energy Physics, Protvino, Russia

GR20/Amaldi10
Warsaw, July 7-13, 2013

Our publications

Bigravity in Kuchar's Hamiltonian formalism. 1. The general case
Vladimir O. Soloviev and Margarita V. Tchichikina
arXiv:1211.6530;

Bigravity in Kuchar's Hamiltonian formalism. 2. The special case
Vladimir O. Soloviev and Margarita V. Tchichikina
arXiv:1302.5096 (2nd version - April 2013)

Contents

History and terminology

- Bimetric theory, N. Rosen (1940)
- f-g gravity, Isham, Salam, Strathdee (1971)
- Strong gravitation, Zumino (1971)
- Bigravity, Damour, Kogan (2002)
- Bimetric gravity, Hassan, R. Rosen (2011)
- ADM Hamiltonian formalism, Arnowitt, Deser, Misner (1962)
- Kuchař's Hamiltonian formalism, Kuchař (1977)

Problems of old massive gravity

- Boulware-Deser ghost (negative kinetic energy)
- Causality problems (two different light cones)
- Nonperturbative solutions for small distances (Wainstein radius)
- No Hamiltonian constraint (arbitrary initial conditions)

S. Deser, A. Waldron (arxiv:1212.5835):

...“*the subject remained moribund until the recent (independent) rediscovery (dRGT, 2011) of the results by Wess-Zumino (1970). This exhumation has, unsurprisingly, generated an immense industry. Our purpose is to re-inter at least one model...*

Bigravity

The bigravity Lagrangian

$$\mathcal{L} = \mathcal{L}^{(f)} + \mathcal{L}^{(g)} - \sqrt{-f}\, U(f_{\mu\nu}, g_{\mu\nu}).$$

is composed of two almost independent parts

$$\mathcal{L}^{(f)} = \frac{1}{16\pi G^{(f)}} \sqrt{-f} f^{\mu\nu} R^{(f)}_{\mu\nu} + \mathcal{L}^{(f)}_M(\psi^A, f_{\mu\nu}),$$

$$\mathcal{L}^{(g)} = \frac{1}{16\pi G^{(g)}} \sqrt{-g} g^{\mu\nu} R^{(g)}_{\mu\nu} + \mathcal{L}^{(g)}_M(\phi^A, g_{\mu\nu}),$$

and the potential of their interaction

$$\sqrt{-f}\, U(f_{\mu\nu}, g_{\mu\nu}) \quad \text{or} \quad \sqrt[4]{fg}\, V(f_{\mu\nu}, g_{\mu\nu}) \quad \text{or} \quad \sqrt{-g}\, W(f_{\mu\nu}, g_{\mu\nu}).$$

de Rham, Gabadadze, Tolley

Both the modern massive gravity and the bigravity (sometimes called as the bimetric gravity) exploit the dRGT potential expressed by means of the matrix square root. Let

$$Y^{\alpha}_{\beta} = g^{\alpha\mu} f_{\mu\beta},$$

then

$$X = \sqrt{Y}, \qquad Y^{\alpha}_{\beta} = X^{\alpha}_{\mu} X^{\mu}_{\beta}.$$

The potential is formed as a linear combination of coefficients of the characteristic polynomial of matrix X^{α}_{β}

$$P(\lambda) = Det(X - \lambda I),$$

i.e. as a linear combination of symmetric polynomials of this matrix eigenvalues

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_2\lambda_4 + \lambda_1\lambda_3,$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2,$$

$$\lambda_1\lambda_2\lambda_3\lambda_4,$$

or by means of the traces of the matrix polynomials,

$$SpX,$$

$$\frac{1}{2}\left((SpX)^2 - SpX^2\right),$$

$$\frac{1}{6}\left((SpX)^3 - 3SpXSpX^2 + 2SpX^3\right),$$

$$\frac{1}{24}\left((SpX)^4 - 6(SpX)^2SpX^2 + 3(SpX^2)^2 + 8SpXSpX^3 - 6SpX^4\right).$$

Statement of the problem

Direct calculations of the Poisson brackets for the dRGT potential are not transparent enough, so the results derived by Hassan-Rosen (2011) and some other authors (including the next speakers) seem to us requiring a further simplification and clarification.

Our plan was to start with a potential of a general form, and then to derive conditions on this potential necessary and (or) sufficient to have the same properties as the dRGT potential. We have found these conditions and they are given below.

Notations

One-parametrical family of spacelike (in both metrics) hypersurfaces is defined by equations

$$X^\alpha = e^\alpha(\tau, x^i),$$

then

$$N^\alpha \equiv \frac{\partial e^\alpha}{\partial \tau}, \qquad e_i^\alpha \equiv \frac{\partial e^\alpha}{\partial x^i}.$$

The induced on hypersurfaces metrics are

$$\gamma_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu, \quad \eta_{ij} = f_{\mu\nu} e_i^\mu e_j^\nu.$$

There are two different fields of unit normals for any hypersurface, $n^\alpha(x^i)$ and $\bar{n}^\alpha(x^i)$ related by equation

$$n^\alpha = u\bar{n}^\alpha + u^i e_i^\alpha,$$

and there are two bases (n^α, e_i^α), $(\bar{n}^\alpha, e_i^\alpha)$ applicable for $3+1$-decompositions of the spacetime vectors and tensors.

The suitable variables are the following

$$u = \frac{1}{\sqrt{-g^{\perp\perp}}}, \qquad u^i = -\frac{g^{\perp i}}{g^{\perp\perp}}.$$

They have a transparent geometric meaning: u is the inverse of a norm (calculated in g-metric) of unit normal vector n^α, constructed according to f-metric, and u^i are projections (in g-metric) of the hypersurface coordinate basis vectors onto n^α:

$$u = \frac{1}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}, \qquad u^i = \frac{g^{\mu\nu} n_\mu e^i_\nu}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}.$$

Two sets of lapses and shifts are related by formulas:

$$\bar{N} = uN, \qquad \bar{N}^i = N^i + u^i N,$$

inverse formulas are

$$u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}.$$

Bigravity Hamiltonian

Let

$$\tilde{U} = \sqrt{\eta} U.$$

$$\begin{aligned} \mathrm{H}_{\mathrm{canonical}} &= \mathrm{H}^{(f)} + \mathrm{H}^{(g)} + \int d^3x N\sqrt{\eta} U \\ &= \int d^3x \left(N\mathcal{H} + N^i\mathcal{H}_i + \bar{N}\bar{\mathcal{H}} + \bar{N}^i\bar{\mathcal{H}}_i + N\tilde{U} \right), \end{aligned}$$

Another form is as follows

$$\mathrm{H}_{\mathrm{canonical}} = \int d^3x \left[N\left(\mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \tilde{U} \right) + N^i \left(\mathcal{H}_i + \bar{\mathcal{H}}_i \right) \right].$$

Primary constraints

$$\pi_N = 0, \quad \pi_{N^i} = 0, \quad \pi_u = 0, \quad \pi_{u^i} = 0.$$

Secondary constraints

$$\mathcal{R} \equiv \mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \tilde{U} = 0,$$

$$\mathcal{R}_i \equiv \mathcal{H}_i + \bar{\mathcal{H}}_i = 0,$$

$$\mathcal{S} \equiv \bar{\mathcal{H}} + \frac{\partial \tilde{U}}{\partial u} = 0, \qquad \mathcal{S}_i \equiv \bar{\mathcal{H}}_i + \frac{\partial \tilde{U}}{\partial u^i} = 0.$$

The Jacobian for $\mathcal{S}, \mathcal{S}_i$ is the Hessian for $\tilde{U}$:

$$\frac{D(\mathcal{S}, \mathcal{S}_i)}{D(u, u^j)} = \left| \frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b} \right| = 0.$$

Dirac brackets

Let the set of second class constraints is denoted as χ_A, $A = 1, .., 6$, where

$$\chi_A = (\pi_i, \mathcal{S}_i)\,,$$

then the following Poisson brackets matrix is nondegenerate

$$||\{\chi_A(x), \chi_B(y)\}|| = \begin{pmatrix} \mathbf{0} & -\mathbf{L}(x)\delta(x,y) \\ \mathbf{L}(x)\delta(x,y) & \mathbf{K}(x,y) \end{pmatrix},$$

and we are able to introduce Dirac brackets:

$$\{F, G\}_D = \{F, G\} - \int dx \int dy \{F, \chi_A(x)\} \mathbf{C}^{AB}(x,y) \{\chi_B(y), G\}.$$

where

$$\mathbf{C}(x,y) = \begin{pmatrix} \mathbf{L}^{-1}(x)\mathbf{K}(x,y)\mathbf{L}^{-1}(y) & \mathbf{L}^{-1}(x)\delta(x,y) \\ -\mathbf{L}^{-1}(x)\delta(x,y) & \mathbf{0} \end{pmatrix}.$$

The most important Dirac bracket is

$$\{\mathcal{S}_x, \mathcal{S}_y\}_D \approx \Theta^i_x \delta_{,i}(x,y) - \Theta^i_y \delta_{,i}(y,x),$$

where

$$\Theta^i \equiv \left(\bar{U}^k \hat{D} \left(\delta^i_k - 2\gamma_{jk} \frac{\partial}{\partial \gamma_{ij}} \right) - \gamma^{ij} \frac{\partial}{\partial u^j} \right) \tilde{U}.$$

First, J. Kluson in 1109.3052 claimed this is nonzero, so DoF$= 5\frac{1}{2}$.
Then, S. Hassan and R. Rosen in 1111.2070 claimed they proved it is zero for the dRGT potential.
So, for our potential $\tilde{U}$ should we believe or prove that Θ^i is zero?
For D. Comelli, F. Nesti and L. Pilo $\Theta^i = 0$ is as an axiom.
We have proved this. But only thanks to D. Comelli, F. Nesti and L. Pilo, who have given in 1302.4447 the reference to work
D. Fairlie and A. Leznov (that time a member of our Department), Solving Monge-Ampere equation, hep-th/940313.

Let us display the necessary and sufficient conditions

$$Q_i^j \equiv 2\eta_{ik}\frac{\partial \tilde{U}}{\partial \eta_{jk}} + 2\gamma_{ik}\frac{\partial \tilde{U}}{\partial \gamma_{jk}} - u^j\frac{\partial \tilde{U}}{\partial u^i} - \delta_i^j \tilde{U} = 0,$$

$$Q^i \equiv 2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ik}} - u^i u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{ik} - u^2\gamma^{ik} - u^i u^k\right)\frac{\partial \tilde{U}}{\partial u^k} = 0.$$

for constraints $\mathcal{R}$, $\mathcal{R}_i$ beeing 1st class:

$$\{\mathcal{R}_{ix}, \mathcal{R}_y\} \approx \mathcal{R}_x \delta_{,i}(x,y) + \frac{\partial}{\partial x^j}\Big(Q_i^j(x)\delta(x,y)\Big),$$

$$\{\mathcal{R}_x, \mathcal{R}_y\} \approx \left(\eta^{ij}\mathcal{R}_j + Q^i\right)_x \delta_{,i}(x,y) - \left(\eta^{ij}\mathcal{R}_j + Q^i\right)_y \delta_{,i}(y,x),$$

$$\{\mathcal{R}_{ix}, \mathcal{R}_{jy}\} = \mathcal{R}_{iy}\delta_{,j}(x,y) + \mathcal{R}_{jx}\delta_{,i}(x,y).$$

Tertiary and quaternary constraints

Tertiary constraint Ω appears from

$$\{\mathcal{S}_x, \mathcal{R}_y\}_D \approx -\Omega_x \delta(x, y).$$

Quaternary constraint Ψ appears from

$$\{\Omega_x, \mathrm{H}\}_D \approx \int d^3y \{\Omega_x, \mathcal{R}_y\}_D N_y \approx \int d^3y \Psi_x \delta(x, y) N_y = 0,$$

Ψ is linear in variable u, as $\mathcal{R} = u\mathcal{S} + \ldots$ and so can be solved for it.

Table of Dirac brackets between constraints

$\{,\}_{\tilde{D}}$	$\pi_u(y)$	$\Psi(y)$	$\Omega(y)$	$\mathcal{S}(y)$	$\mathcal{R}(y)$	$\mathcal{R}_j(y)$
$\pi_u(x)$ (primary)	0	$\neq 0$	$-\hat{\Theta} = 0$	0	≈ 0	0
$\Psi(x)$ (quaternary)	$\neq 0$					
$\Omega(x)$ (tertiary)	$\hat{\Theta} = 0$			$\neq 0$	$\Psi \approx 0$	≈ 0
$\mathcal{S}(x)$(*secondary*)	0		$\neq 0$	$\hat{\Theta} = 0$	≈ 0	≈ 0
$\mathcal{R}(x)$ (secondary)	≈ 0		$-\Psi \approx 0$	≈ 0	≈ 0	≈ 0
$\mathcal{R}_i(x)$ (secondary)	0		≈ 0	≈ 0	≈ 0	≈ 0

Results: axioms for the potential

1. We have a differentiable function $\tilde{U} = \tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij})$.
2. Diffeomorphism invariance requires

$$
\begin{aligned}
2\eta_{ik}\frac{\partial \tilde{U}}{\partial \eta_{jk}} + 2\gamma_{ik}\frac{\partial \tilde{U}}{\partial \gamma_{jk}} - u^j \frac{\partial \tilde{U}}{\partial u^i} - \delta_i^j \tilde{U} &= 0, \\
2u^j \gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ik}} - u^i u \frac{\partial \tilde{U}}{\partial u} + \left(\eta^{ik} - u^2\gamma^{ik} - u^i u^k\right)\frac{\partial \tilde{U}}{\partial u^k} &= 0.
\end{aligned}
$$

3. The big Hessian matrix is to be degenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}\right| = 0, \qquad u^a = (u, u^i).
$$

4. The small Hessian matrix is to be nondegenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j}\right| \neq 0, \qquad i = 1, 2, 3.
$$